

On a classification of minimal cubic cones in \mathbb{R}^n

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ABSTRACT. We establish a classification of cubic minimal cones in case of the so-called radial eigencubics. Our principal result states that any radial eigencubic is either a member of the infinite family of eigencubics of Clifford type, or belongs to one of 18 exceptional families. We prove that at least 12 of the 18 families are non-empty and study their algebraic structure. We also establish that any radial eigencubic satisfies the trace identity $\det \text{Hess}^3(f) = \alpha f$ for the Hessian matrix of f , where $\alpha \in \mathbb{R}$. Another result of the paper is a correspondence between radial eigencubics and isoparametric hypersurfaces with four principal curvatures.

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1. Preliminaries and the main results

1.1. Introduction. In 1969, Bombieri, De Giorgi and Giusti [BGG] found the first non-affine entire solution of the minimal surface equation

$$(1 + |\nabla u|^2) \Delta u - \sum_{i,j=1}^{n-1} u_{x_i x_j} u_{x_i} u_{x_j} = 0 \quad (1.1)$$

Because of its geometric significance, the minimal surface equation (1.1) and, especially, Bernstein's problem on the existence of non-affine entire solutions of (1.1), have historically attracted perhaps more interest than any other quasilinear elliptic equation. We refer to [M], [MMM], [Ni], [Os], [S1], and the references therein for a detailed discussion of the history of the solution of Bernstein's problem. Although many non-affine examples of entire solutions of (1.1) for $n \geq 9$ were shown to exist (see, for instance, [S2], [SS]), no explicit examples have been constructed. L. Simon [S2] established that for $n = 9$ all entire solutions of (1.1) are of polynomial growth and it is a long-standing conjecture that this property holds in general [BG], [Os]. Even a simpler question [S1], [M], whether or not there exists a solution of (1.1) which is actually a polynomial in x_i , is still unanswered.

These questions prompt one to study algebraic minimal hypersurfaces, and, in particular, algebraic minimal cones. The latter occur naturally as singular ‘blow-ups’ of entire solutions of (1.1) at infinity; for example, the seven-dimensional Simon’s cone $\{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 = |y|^2\}$ played an important role in the solution of Bernstein’s problem [F1], [SJ] and in the constructing of non-affine examples by Bombieri, de Giorgi and Giusti [BGG]. We mention also a recent appearance of algebraic minimal hypersurfaces as selfsimilar solutions of the mean curvature flow in codimension one [Sm]. Note also that any progress in algebraic minimal cones leads to a better understanding of algebraic aspects of minimal submanifolds of codimension one in the unit spheres because of the well-known correspondence between these objects.

Minimal cones of lower degrees were classified by Hsiang [H]: the only first degree minimal cones are hyperplanes in \mathbb{R}^n , and the only (up to a congruence in \mathbb{R}^n) quadratic minimal cones are given by the zero-locus $g^{-1}(0)$ of the quadratic forms

$$g(x) = (n - p - 1)(x_1^2 + \dots + x_p^2) - (p - 1)(x_{p+1}^2 + \dots + x_n^2), \quad 2 \leq p \leq n - 1. \quad (1.2)$$

On the other hand, a classification (and even construction) of algebraic minimal cones of degree higher than three remains a long-standing difficult problem [S1], [H], [F], [Os]. The lack of ‘canonical’ normal forms for higher degree polynomials makes a classification of algebraic minimal cones, at first sight, defeating. On the other hand, a close analysis of the available examples of minimal cubic cones, see e.g. [Ta], [H1], [HL], [L], reveals that these cones have a rather distinguished algebraic structure which, in some content, resembles that of isoparametric hypersurfaces in the spheres. In [H], Hsiang began developing a systematic approach to study real algebraic minimal submanifolds of degree higher than two and by using the geometric invariant theory constructed new examples of non-homogeneous minimal cubic cones in \mathbb{R}^9 and \mathbb{R}^{15} . According to Hsiang, the study of real algebraic minimal cones is equivalent to a classification of polynomial solutions $f = f(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$ of the following congruence:

$$L(f) \equiv 0 \pmod{f}, \quad (1.3)$$

where

$$L(f) = |\nabla f|^2 \Delta f - \sum_{i,j=1}^n f_{x_i} f_{x_j} f_{x_i x_j}$$

is the normalized mean curvature operator and (1.3) is understood in the usual sense, i.e. $L(f)$ is divisible by f in the polynomial ring $\mathbb{R}[x_1, \dots, x_n]$. Observe, that (1.3) geometrically means that the zero-locus $f^{-1}(0)$ has zero mean curvature everywhere where the gradient $\nabla f \neq 0$.

A polynomial solution $f \not\equiv 0$ of (1.3) is called an *eigenfunction* of L . The ratio $L(f)/f$ (which is obviously a polynomial in x) is called the *weight* of an eigenfunction f . An eigenfunction f which is a cubic homogeneous polynomial is also called an *eigencubic*.

Remark 1.1. For geometric reasons, we make no distinction between two eigenfunctions f_1 and f_2 which give rise to two congruent cones $f_1^{-1}(0)$ and $f_2^{-1}(0)$; such eigenfunctions will also be called *congruent*. It follows from the real Nullstellensatz [Mi] (see also [T2, Proposition 2.4]) that two irreducible homogeneous cubic polynomials f_1 and f_2 are congruent if and only if there exists an orthogonal endomorphism of $U \in O(\mathbb{R}^n)$ and a constant $c \in \mathbb{R}$, $c \neq 0$, such that $f_1(x) = cf_2(Ux)$.

In [H], Hsiang observes that all available cubic minimal cones arise as solutions of the following non-linear equation:

$$L(f) = \lambda |x|^2 f, \quad \lambda \in \mathbb{R}, \quad (1.4)$$

and poses the problem to determine all solutions of (1.4) up to congruence in \mathbb{R}^n . We call the solutions of (1.4) *radial eigencubics*.

It is the purpose of the present paper to provide a general framework for a classification of radial eigencubics. We prove that any radial eigencubic f is a harmonic polynomial and associate to it a pair (n_1, n_2) of non-negative integers, called the type of f . We show that the type is a

congruence invariant of f and establish that n_1 can be recovered by the following remarkable trace identity:

$$\text{tr Hess}^3(f) = 3(n_1 - 1)\lambda f,$$

where $\text{Hess}(f)$ is the Hessian matrix of f and the constant factor λ is the same as in (1.4). The principal result of the paper states that any radial eigencubic is either a member of the infinite family of eigencubics of Clifford type introduced and classified recently in [T2], or belongs to one of 18 exceptional families which types (n_1, n_2) and ambient dimensions n listed in Table 1 below. We also establish that at least 12 of the 18 families are non-empty and provide examples of radial eigencubics for each realizable family.

n_1	2	3	5	9	0	1	2	4	0	1	5	9	0	1	3	1	3	7
n_2	0	0	0	0	5	5	5	5	8	8	8	8	14	14	14	26	26	26
n	5	8	14	26	9	12	15	21	15	18	30	42	27	30	36	54	60	72
							?				?	?			?		?	?

TABLE 1. Exceptional eigencubics: ? stands for the unsettled cases.

As was already mentioned, a classification of general radial eigencubics closely resembles that of isoparametric hypersurfaces with four principal curvatures. The isoparametric hypersurfaces have been intensively studied for several decades now and, at the present, a complete classification is available for all but for four exceptional isoparametric families, see [Mu1], [Mu2], [OT1], [OT2], [A], [St], [C1], [C2]. It would be interesting to work out an explicit correspondence between these theories. We mention that, in one direction, a theorem of Nomizu [No] states that each focal variety of an isoparametric hypersurface is a minimal submanifold of the ambient unit sphere. On the other hand, in the present paper we show that, in the other direction, to any non-isoparametric radial eigencubic one can associate an isoparametric hypersurface with four principal curvatures. Combining the latter correspondence with a deep characterization of isoparametric quartics obtained recently by T. Cecil, Q.S.Chi and G. Jensen [CCC], and by S. Immerwoll [Im], we obtain an obstruction to the existence of some exceptional families of radial eigencubics.

Remark 1.2. We would like to emphasize that, in general, real algebraic minimal cones have a much more rich structure than isoparametric hypersurfaces. Indeed, in the former case, the examples constructed recently in [T2] show that there exist irreducible minimal cones of arbitrary high degree, while the well-known theorem of Münzner [Mu1] allows the defining polynomials of isoparametric hypersurfaces to be only of degrees $g = 1, 2, 3, 4$ and 6.

In section 1.4 below we consider our results in more detail. First we recall some basic facts about eigencubics of Clifford type and Cartan's isoparametric polynomials.

1.2. Eigencubics of Clifford type. A system of symmetric endomorphisms $\mathcal{A} = \{A_i\}_{0 \leq i \leq q}$ of \mathbb{R}^{2m} is called a symmetric Clifford system [Hu], [C2], [BW], equivalently $\mathcal{A} \in \text{Cliff}(\mathbb{R}^{2m}, q)$ if

$$A_i A_j + A_j A_i = 2\delta_{ij} \cdot \mathbf{1}_{\mathbb{R}^{2m}},$$

where $\mathbf{1}_V$ stands for the identity operator in a linear space V . To any symmetric Clifford system $\mathcal{A} \in \text{Cliff}(\mathbb{R}^{2m}, q)$ with two distinguished elements $A_0, A_1 \in \mathcal{A}$ one can associate an orthogonal eigen-decomposition $\mathbb{R}^{2m} = \mathbb{R}^m \oplus \mathbb{R}^m$ such that

$$A_0 = \begin{pmatrix} \mathbf{1}_{\mathbb{R}^m} & 0 \\ 0 & -\mathbf{1}_{\mathbb{R}^m} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \mathbf{1}_{\mathbb{R}^m} \\ \mathbf{1}_{\mathbb{R}^m} & 0 \end{pmatrix}, \quad A_i = \begin{pmatrix} 0 & P_i \\ P_i^\top & 0 \end{pmatrix}. \quad (1.5)$$

where the skew-symmetric transformations P_1, \dots, P_{q-1} satisfy $P_i P_j + P_j P_i = -2\delta_{ij}$. This determines a representation $\{P_1, \dots, P_{q-1}\}$ of the Clifford algebra Cl_{q-1} on \mathbb{R}^m [Hu], [BW]. Conversely, any representation of the Clifford algebra Cl_{q-1} on \mathbb{R}^m induces a symmetric Clifford

system by virtue of (1.5). It follows from the representation theory of Clifford algebras that the class $\text{Cliff}(\mathbb{R}^{2m}, q)$ is non-empty if and only if

$$q \leq \rho(m), \quad (1.6)$$

where the Hurwitz-Radon function ρ is defined by

$$\rho(m) = 8a + 2^b, \quad \text{if } m = 2^{4a+b} \cdot \text{odd}, \quad 0 \leq b \leq 3. \quad (1.7)$$

Two symmetric Clifford systems $\mathcal{A} \in \text{Cliff}(\mathbb{R}^{2m}, q)$ and $\mathcal{B} \in \text{Cliff}(\mathbb{R}^{2m'}, q')$ are called *geometrically equivalent*, if $q = q'$, $m = m'$, and there exist orthogonal endomorphisms $U \in O(\mathbb{R}^{2m})$ and $u \in O(\mathbb{R}^{q+1})$ such that

$$A_{uz} = U^\top B_z U, \quad \forall z \in \mathbb{R}^{q+1},$$

where $A_z = \sum_{i=0}^q z_i A_i$, $B_z = \sum_{i=0}^q z_i B_i$. Then the cardinality $\kappa(m, q)$ of the quotient set of $\text{Cliff}(\mathbb{R}^{2m}, q)$ with respect to the geometric equivalence is equal to 1 for $q = 0$ and is determined for $q \geq 1$ by the following formula (see also [C2, § 4.7]):

$$\kappa(m, q) = \begin{cases} 0, & \text{if } \delta(q) \nmid m; \\ 1, & \text{if } \delta(q) \mid m \text{ and } q \not\equiv 0 \pmod{4}; \\ \lfloor \frac{m}{2\delta(q)} \rfloor + 1, & \text{if } \delta(q) \mid m \text{ and } q \equiv 0 \pmod{4}, \end{cases} \quad (1.8)$$

where $\lfloor x \rfloor$ is the integer part of x and $\delta(q) = \min\{2^k : \rho(2^k) \geq q\}$, or equivalently by the following table [BW, p. 156]:

q	1	2	3	4	5	6	7	8	...	k
$\delta(q)$	1	2	4	4	8	8	8	8	...	$16\delta(k-8)$

In [T2] we associated to a Clifford symmetric system $\mathcal{A} \in \text{Cliff}(\mathbb{R}^{2m}, q)$ the cubic form

$$C_{\mathcal{A}}(x) := \sum_{i=0}^q \langle y, A_i y \rangle x_{i+1}, \quad y = (x_{q+2}, \dots, x_{q+1+2m}) \in \mathbb{R}^{2m}, \quad (1.9)$$

and proved that $C_{\mathcal{A}}$ is a radial eigencubic in \mathbb{R}^{2m+q+1} .

Definition. An arbitrary radial eigencubic is said to be of *Clifford type* if it is congruent to some $C_{\mathcal{A}}$. Otherwise it is called an *exceptional* radial eigencubics.

We also proved in [T2] that the congruence classes of eigencubics of Clifford type are in one-to-one correspondence with the equivalence classes of geometrically equivalent Clifford systems which, in view of the remarks made above, yields a complete classification of eigencubics of Clifford type.

1.3. The Cartan isoparametric polynomials. In [Car], E. Cartan proved that, up to congruence, the only *irreducible* cubic polynomial solutions of the isoparametric system

$$|\nabla f|^2 = 9x^4, \quad \Delta f = 0 \quad (1.10)$$

are the following four polynomials:

$$\theta_\ell = x_1^3 + \frac{3x_1}{2}(|z_1|^2 + |z_2|^2 - 2|z_3|^2 - 2x_2^2) + \frac{3\sqrt{3}}{2}[x_2(|z_1|^2 - |z_2|^2) + \text{Re } z_1 z_2 z_3], \quad (1.11)$$

where $z_k = (x_{k\ell-\ell+3}, \dots, x_{k\ell+2}) \in \mathbb{R}^\ell = \mathbb{F}_\ell$, and \mathbb{F}_ℓ denote the division algebra of dimension ℓ (over reals): $\mathbb{F}_1 = \mathbb{R}$ (reals), $\mathbb{F}_2 = \mathbb{C}$ (complexes), $\mathbb{F}_4 = \mathbb{H}$ (quaternions) and $\mathbb{F}_8 = \mathbb{O}$ (octonions). The real part in (1.11) should be understood for a general \mathbb{F}_ℓ as

$$\text{Re } z_1 z_2 z_3 = \frac{1}{2}((z_1 z_2) z_3 + \bar{z}_3(\bar{z}_2 \bar{z}_1)) = \frac{1}{2}(z_1(z_2 z_3) + (\bar{z}_3 \bar{z}_2) \bar{z}_1), \quad (1.12)$$

(observe that the real part is associative, see also Lemma 15.12 in [Ad]).

It follows from (1.11) that the Cartan isoparametric cubics are well-defined only if the ambient dimension $n \in \{5, 8, 14, 26\}$. Moreover, in virtue of (1.10)

$$L(f) = -\frac{1}{2}\langle \nabla|\nabla f|^2, \nabla f \rangle = -18\langle x, \nabla f \rangle = -54x^2f,$$

hence any Cartan polynomial is also a radial eigencubic. It is easily seen that any θ_ℓ is in fact an exceptional eigencubic. Indeed, we note that the squared norm of the gradient is a congruence invariant and $|\nabla\theta_\ell|^2 = 9x^4$, while (1.9) yields that the squared norm of the gradient of an eigencubic of Clifford type is at most quadratic in some variables.

A crucial role in our further analysis will play the following generalization of the Cartan theorem obtained by the author in [T1].

The Eiconal Cubic Theorem. *Let $f(x)$ be a cubic polynomial solution of the first equation in (1.10) alone. Then f is either reducible and congruent to $x_n(x_n^2 - 3x_1^2 - \dots - 3x_{n-1}^2)$, or irreducible and congruent to some Cartan polynomial $\theta_\ell(x)$.*

1.4. Main results. As the first step we obtain the following characterization of general radial eigencubics.

Theorem 1. *Any radial eigencubic in \mathbb{R}^n is a harmonic function.*

Remark 1.3. Observe, however, that there are (non-radial) eigencubics which are non-harmonic, for example, $f = x_1g(x)$ with g given by (1.2). All such non-harmonic eigenfunctions are reducible, so it would be interesting to know whether there exist *irreducible* non-harmonic eigencubics.

Our next step is to establish (Proposition 3.1) that given a radial eigencubic f in \mathbb{R}^n one can associate the orthogonal coordinates $\mathbb{R}^n = \text{span}(e_n) \oplus V_1 \oplus V_2 \oplus V_3$ in which f takes the so-called *normal form*

$$f = x_n^3 + \phi x_n + \psi \equiv x_n^3 - \frac{3}{2}x_n(2\xi^2 + \eta^2 - \zeta^2) + \psi_{111} + \psi_{102} + \psi_{012} + \psi_{030}, \quad (1.13)$$

where $x = (\xi, \eta, \zeta, x_n) \in \mathbb{R}^n$, $\xi \in V_1$, $\eta \in V_2$, $\zeta \in V_3$ and ψ_{ijk} denotes a cubic form of homogeneous class $\xi^i \otimes \eta^j \otimes \zeta^k$. (Here and in what follows, if no ambiguity possible, we abuse the norm notation by writing, e.g., ξ^2 for $|\xi|^2$). In addition, the harmonicity of f yields the following restrictions:

$$n_3 = 2n_1 + n_2 - 2, \quad n = 3n_1 + 2n_2 - 1. \quad (1.14)$$

Definition. The pair (n_1, n_2) is called the *type* of the normal form.

Thus a very natural question appears from the very beginning: whether the type (n_1, n_2) has an invariant meaning? We answer this question in positive, but what is more important, we establish the following remarkable trace identity for determining of the dimension n_1 .

Theorem 2. *Let f be a radial eigencubic in the normal form (1.13). Then the associated dimensions $n_i = \dim V_i$, $i = 1, 2, 3$, do not depend on a particular choice of the normal form of f and can be recovered by virtue of the cubic trace formula*

$$n_1 = 1 + \frac{x^2 \text{tr Hess}^3(f)}{3L(f)}, \quad (1.15)$$

and relations $n_2 = \frac{1}{2}(n - 3n_1 + 1)$, $n_3 = 2n_1 + n_2 - 2$, where $\text{Hess}(f)$ is the Hessian matrix of f .

We emphasize that the ratio in (1.15) is an integer number.

In [T2], we established the cubic trace identity for eigencubics of Clifford type. In the present paper, we extend the cubic trace identity to the general radial eigencubics. Our argument is heavily based on the characterization of exceptional eigencubics by means of the ψ_{030} -term in (1.13) which we now describe. Let us assume that f be a radial eigencubic written in the normal form. First we

show in Proposition 3.3 that the combination $\psi_{111} + \frac{1}{\sqrt{3}}\psi_{102}$ induces a symmetric Clifford system in $\text{Cliff}(\mathbb{R}^{2(n_1+n_2-1)}, n_1 - 1)$ which immediately yields by virtue of (1.6) the inequality

$$n_1 - 1 \leq \rho(n_2 + n_1 - 1). \quad (1.16)$$

Next we prove (Proposition 3.2) that the cubic form ψ_{030} in (1.13) satisfies an eiconal type equation $|\nabla\psi_{030}(\eta)|^2 = \frac{9}{2}\eta^2$, $\eta \in \mathbb{R}^{n_2}$. Combining these observations and some further properties of the normal form, we are able to prove the following important characterization of the Clifford eigencubics.

Theorem 3. *A radial eigencubic f is of Clifford type if and only if for any particular choice of its normal form (1.13), the component $\psi_{030} \not\equiv 0$ and reducible.*

In particular, by combining Theorem 3 with the Eiconal Cubic Theorem above, one obtains that for any exceptional eigencubic there holds $n_2 \in \{0, 5, 8, 14, 26\}$. Then by using some special properties of the Hurwitz-Radon function ρ one is able to show that there exists only finitely many pairs (n_1, n_2) satisfying the above inclusion and (1.16). This yields the finiteness of the number of types of exceptional eigencubics. In fact, we have the the following criterion.

Theorem 4. *Let f be a radial eigencubic in \mathbb{R}^n . Then the following statements are equivalent:*

- (a) *f is an exceptional radial eigencubic;*
- (b) *for any choice of the normal form (1.13), the form ψ_{030} is either irreducible or identically zero;*
- (c) *$n_2 \in \{0, 5, 8, 14, 26\}$ and the quadratic form*

$$\sigma_2(f) := -\frac{1}{3\lambda}\text{Hess}^2(f), \quad \text{where } L(f) = \lambda x^2 f,$$

has a single eigenvalue.

The only possible types of exceptional eigencubics are those displayed in Table 2 below.

In the remaining part of the paper we investigate which of the 23 pairs (n_1, n_2) in Table 2 are indeed realizable as the types of exceptional eigencubics. Below we summarize the corresponding results.

- (i) For $n_2 = 0$ all types $(\ell + 1, 0)$, $\ell = 1, 2, 4, 8$, are realizable. For each ℓ , there is exactly one congruence class of exceptional eigencubics of type $(\ell + 1, 0)$ represented by the Cartan polynomial θ_ℓ .
- (ii) For $n_1 = 0$ the only three types $(0, 5)$, $(0, 8)$ and $(0, 14)$ are realizable.
- (iii) For $n_1 = 1$ then four types $(1, 5)$, $(1, 8)$, $(1, 14)$ and $(1, 26)$ are realizable and in each case there is exactly one congruence class of exceptional eigencubics.
- (iv) There is an exceptional eigencubic of type of type $(4, 5)$.
- (v) The types $(2, 8)$, $(2, 14)$, $(2, 26)$ and $(3, 8)$ are not realizable.

Thus, it remains unsettled the six exceptional pairs: $(2, 5)$, $(5, 8)$, $(9, 8)$, $(3, 14)$, $(3, 26)$, $(7, 26)$.

To obtain the non-existence result (v) we develop a correspondence between general radial eigencubics with $n_2 \neq 0$ and isoparametric quartic polynomials which can be described as follows. Recall that a hypersurface in the unit sphere in \mathbb{R}^n is called *isoparametric* if it has constant principal curvatures [Th1], [C1]. A celebrated theorem of Münzner [Mu1] states that any isoparametric hypersurface is algebraic and its defining polynomial h is homogeneous of degree $g = 1, 2, 3, 4$ or 6, where g is the number of distinct principal curvatures. Moreover, if $g = 4$ then, suitably normalized, h satisfies the system Münzner-Cartan differential equations (cf. with (1.10 above)

$$|\nabla h|^2 = 16x^6, \quad \Delta h = 8(m_2 - m_1)x^2, \quad x \in \mathbb{R}^n, \quad (1.17)$$

where m_i are the multiplicities of the maximal and minimal principal curvature of M , $m_1 + m_2 = \frac{n-2}{2}$. Let $\text{Isop}(m_1, m_2)$ denote the class of all quartic polynomials satisfying (1.17). Then each

$h \in \text{Isop}(m_1, m_2)$ with $m_1, m_2 \geq 1$ gives rise to a family of isoparametric hypersurfaces

$$M_c = \{x \in \mathbb{S}^{n-1} \subset \mathbb{R}^n \mid h(x) = c\}, \quad c \in (-1, 1),$$

see for instance [C2, p. 96-97]. In this case m_1 and m_2 are, up to a permutation, the multiplicities of the maximal and minimal principal curvature of the hypersurface M_c and

$$n - 2 = 2(m_1 + m_2). \quad (1.18)$$

Theorem 5. *Let f be any radial eigencubic in \mathbb{R}^n of type (n_1, n_2) , $n_2 \neq 0$, and normalized by $\lambda = -8$ in (1.4). Then f can be written in some orthogonal coordinates in the degenerate form*

$$f = (u^2 - v^2)x_n + a(u, w) + b(y, w) + c(u, y, w), \quad (1.19)$$

where $u = (x_1, \dots, x_m)$, $v = (x_{m+1}, \dots, x_{2m})$, $w = (x_{2m+1}, \dots, x_{n-1})$, and the cubic forms $a \in u \otimes w^2$, $b \in v \otimes w^2$, $c \in u \otimes v \otimes w$. Moreover, the quartic polynomials

$$\begin{aligned} h_0(u, v) &:= (u^2 + v^2)^2 - 2c_w^2 \in \text{Isop}(n_1 - 1, m - n_1), \\ h_1(u, v) &:= -u^4 + 6u^2v^2 - v^4 - 2c_w^2 \in \text{Isop}(n_1, m - n_1 - 1). \end{aligned}$$

If f is in addition an exceptional eigencubic then $n_2 = 3\ell + 2$, $\ell \in \{1, 2, 4, 8\}$ and $m = \ell + n_1 + 1$.

By using a recent classification result of T. Cecil, Q.S.Chi and G. Jensen [CCC], and S. Immerwoll [Im], and Theorem 5, one can deduce the nonexistence of types mentioned in (v).

The paper is organized as follows. In section 2 we prove Theorem 1 and in section 3 we establish the normal representation (1.13). In Proposition 3.3 we exhibit a hidden Clifford structure associated with any radial eigencubic and prove (1.16). In Proposition 3.4 we obtain a complete classification of radial eigencubics with $n_2 = 0$ mentioned in the item (i) above. The proofs of Theorem 2, Theorem 3 and Theorem 4 will be given in sections 4 and 5. In section 6 we establish the classification results (ii)–(iv) and also review some examples of exceptional radial eigencubics and outline some their aspects. In section 7 we prove Theorem 5 and the non-existence result (v).

Notation. We use the standard convention that f_ξ denotes the vector-column of partial derivatives f_{ξ_i} and $f_{\xi\eta}$ stands for the Jacobian matrix with entries $f_{\xi_i\eta_j}$, etc. By $\Delta_\xi f = \text{tr } f_{\xi\xi}$ we denote the Laplacian with respect to ξ . We suppress the variable notation for the full gradient ∇f , the Hessian matrix $\text{Hess} f$ and the full Laplacian Δf . In what follows, if no ambiguity possible, we abuse the norm notation by writing, e.g., ξ^2 for $|\xi|^2$. The bar notation \bar{x} is usually used

2. The harmonicity of radial eigencubics

We begin with treating the normal form of a radial eigencubic. To this end, let us consider an arbitrary radial eigencubic f and let $x_0 \in \mathbb{S}^{n-1}$ be a maximum point of f on the unite sphere \mathbb{S}^{n-1} . It is well known and easily verified that in any orthogonal coordinates with x_0 chosen to be the n -th basis vector, f expands as follows:

$$f(x) = cx_n^3 + x_n\phi(\bar{x}) + \psi(\bar{x}), \quad \bar{x} = (x_1, \dots, x_{n-1}). \quad (2.1)$$

We shall refer to (2.1) as the *normal form* of f . Using the freedom to scale f , we can ensure that $c = 1$. Rewrite the definition of radial eigencubic as follows:

$$L(f) = 18\alpha x^2 f, \quad (2.2)$$

where the factor $\lambda(f) = 18\alpha$ is chosen for the further convenience. Then, identifying the coefficients of x_n^i , $0 \leq i \leq 5$, in (2.2) we arrive at the following system

$$\Delta_{\bar{x}}\phi = 2\alpha, \quad (2.3)$$

$$\Delta_{\bar{x}}\psi = 0, \quad (2.4)$$

$$\phi_{\bar{x}}^2 \Delta_{\bar{x}}\phi - \phi_{\bar{x}}\phi_{\bar{x}\bar{x}}\phi_{\bar{x}} = 6\alpha(\phi + 3\bar{x}^2), \quad (2.5)$$

$$2\phi_{\bar{x}}^\top \phi_{\bar{x}\bar{x}}\psi_{\bar{x}} + \phi_{\bar{x}}^\top \psi_{\bar{x}\bar{x}}\phi_{\bar{x}} - (6 + 4\alpha)\phi_{\bar{x}}^\top \psi_{\bar{x}} + 18\alpha\psi = 0, \quad (2.6)$$

$$2(3 + \alpha)\psi_{\bar{x}}^2 - \psi_{\bar{x}}^\top \phi_{\bar{x}\bar{x}}\psi_{\bar{x}} - 2\phi_{\bar{x}}^\top \psi_{\bar{x}\bar{x}}\psi_{\bar{x}} = 2\phi(9\alpha\bar{x}^2 + \phi_{\bar{x}}^2 - \alpha\phi), \quad (2.7)$$

$$2\phi\phi_{\bar{x}}^\top \psi_{\bar{x}} + 18\alpha\bar{x}^2\psi + \psi_{\bar{x}}^\top \psi_{\bar{x}\bar{x}}\psi_{\bar{x}} = 0. \quad (2.8)$$

We may choose orthogonal coordinates in \mathbb{R}^n such that the quadratic form ϕ becomes diagonal, say $\phi(x) = \sum_{i=1}^{n-1} \phi_i x_i^2$. Then (2.3) yields

$$\sum_{i=1}^{n-1} \phi_i = \alpha. \quad (2.9)$$

By expanding (2.5), we see that each eigenvalue ϕ_i satisfies the equation

$$\chi_\alpha(t) := 4t^3 - 4\alpha t^2 + 3\alpha t + 9\alpha = 0. \quad (2.10)$$

First notice that we can always assume that $\alpha \neq 0$ because otherwise (2.10) yields $\chi_0 \equiv 4t^4$, hence $\phi_i = 0$ for all i and thus $\phi \equiv 0$. But the latter implies $\psi_{\bar{x}}^2 = 0$ by virtue of (2.7), hence $\psi \equiv 0$. This yields $f = x_n^3$, i.e. $n = 1$, a contradiction.

Thus, assuming $\alpha \neq 0$, we denote by t_i , $1 \leq i \leq \nu(\alpha)$, all distinct *real* roots of (2.10). Since (2.10) is a cubic equation with real coefficients, one has $1 \leq \nu(\alpha) \leq 3$. Regarding t_i as an eigenvalue of ϕ , let V_i denote the corresponding eigenspace (V_i may be null-dimensional). Then

$$\mathbb{R}^n = \text{span}(e_n) \oplus V, \quad V = \bigoplus_{i=1}^{\nu(\alpha)} V_i. \quad (2.11)$$

From (2.9) we infer the following constraints on the dimensions $n_i = \dim V_i$:

$$\sum_{i=1}^{\nu(\alpha)} t_i n_i = \alpha, \quad \sum_{i=1}^{\nu(\alpha)} n_i = n - 1. \quad (2.12)$$

Now we are going to specify the algebraic structure of the cubic form ψ . Note that the eigen decomposition (2.11) extends to the tensor products, thus we have for the cubic forms:

$$V^{*\otimes 3} = \bigoplus_{|q|=3} V^{*\otimes q}, \quad V^{*\otimes q} := \bigotimes_{i=1}^{\nu(\alpha)} V_i^{*q_i},$$

where $q = (q_1, \dots, q_{\nu(\alpha)})$ and $|q| = q_1 + \dots + q_{\nu(\alpha)} = 3$. Write $\psi = \sum_{|q|=3} \psi_q$ according to the above decomposition.

Lemma 2.1. *In the above notation, let*

$$R_q := \frac{9\alpha}{2} + \left(\sum_{k=1}^{\nu(\alpha)} t_k q_k \right)^2 + \sum_{k=1}^{\nu(\alpha)} t_k^2 q_k - (3 + 2\alpha) \sum_{k=1}^{\nu(\alpha)} t_k q_k. \quad (2.13)$$

If $R_q \neq 0$ for some q , $|q| = 3$, then the corresponding homogeneous component ψ_q is zero. In other words, ψ is completely determined by the homogeneous components ψ_q whose indices q satisfy $R_q = 0$.

PROOF. By virtue of the Euler homogeneous function theorem,

$$\begin{aligned}\phi_{\bar{x}}^\top \phi_{\bar{x}\bar{x}}(\psi_q)_{\bar{x}} &= 4\psi_q \sum_{k=1}^{\nu(\alpha)} t_k^2 q_k, \\ \phi_{\bar{x}}^\top (\psi_q)_{\bar{x}\bar{x}} \phi_{\bar{x}} &= 4\psi_q \left(\left(\sum_{k=1}^{\nu(\alpha)} t_k q_k \right)^2 - \sum_{k=1}^{\nu(\alpha)} t_k^2 q_k \right), \\ \phi_{\bar{x}}^\top (\psi_q)_{\bar{x}} &= 2\psi_q \sum_{k=1}^3 t_k q_k,\end{aligned}$$

hence (2.6) yields

$$\sum_q R_q \psi_q = 0. \quad (2.14)$$

Since the non-zero components ψ_q are linear independent we get the required conclusion. \square

Lemma 2.2. *If f is a radial eigencubic of dimension $n \geq 2$ then equation (2.10) must have three distinct real roots, i.e. $\nu(\alpha) = 3$. In particular, the discriminant of χ_α is nonzero.*

PROOF. To prove the theorem we shall argue by contradiction and assume that $\nu(\alpha) \leq 2$. This holds only if either (i) all roots ϕ_i are real but the discriminant of $\chi_\alpha(t)$ is zero, or (ii) $\chi_\alpha(t)$ has a pair of conjugate complex roots.

First consider (i). We have for the discriminant (see, for example, [Wa])

$$\mathcal{D}(\chi_\alpha) = 144\alpha^2(17\alpha^2 - 57\alpha - 243).$$

Except for the trivial case $\alpha = 0$, the discriminant vanishes only for $\alpha_\pm := \frac{57 \pm 39\sqrt{13}}{34}$. Since analysis of the two numbers is similar we treat only α^+ . For this value, (2.10) has two distinct roots $t_1 = \frac{3-6\sqrt{13}}{17}$ and $t_2 = \frac{3+3\sqrt{13}}{4}$, the latter of multiplicity two. Thus $\nu(\alpha^+) = 2$ and by virtue of (2.3), $t_1 n_1 + t_2 n_2 = \alpha^+$. A unique integer solution of the latter equation is easily found to be $(n_1, n_2) = (1, 2)$, hence, in view of (2.12), the total dimension $n = 4$. Choose $V_1 = \text{span}(e_1)$ and $V_2 = \text{span}(e_2, e_3)$ so that

$$\phi = t_1 x_1^2 + t_2 (x_2^2 + x_3^2). \quad (2.15)$$

In order to determine ψ , we apply Lemma 2.1. A direct examination of (2.13) shows that among the R_q -coefficients with $q = (i, 3-i)$, $0 \leq i \leq 3$, there is only one zero coefficient, namely $R_{1,2} = 0$. Thus, $\psi \equiv \psi_{12}$, i.e. ψ is linear in x_1 and bilinear in (x_2, x_3) . By (2.4), $\Delta_x \psi = 0$, hence ψ is congruent to the form

$$\psi = b x_1 x_2 x_3, \quad b \in \mathbb{R}. \quad (2.16)$$

Applying the explicit form of ϕ , we get

$$2\phi \phi_{\bar{x}}^\top \psi_{\bar{x}} = 4\phi b (t_1 + 2t_2) x_1 x_2 x_3 = 4b \alpha^+ \phi x_1 x_2 x_3,$$

and

$$18\alpha^+ \bar{x}^2 \psi + \psi_{\bar{x}}^\top \psi_{\bar{x}\bar{x}} \psi_{\bar{x}} = b(18\alpha^+ + 2b^2)(x_1^2 + x_2^2 + x_3^2) x_1 x_2 x_3.$$

Therefore, (2.8) yields that either $b = 0$ or

$$4\alpha^+ \phi = (18\alpha^+ + 2b^2)(x_1^2 + x_2^2 + x_3^2).$$

The latter relation impossible because (2.15) and $t_1 \neq t_2$. Thus $b = 0$. But this implies by virtue of (2.16) that $\psi \equiv 0$ and by (2.7), $\phi = 0$, so the contradiction follows.

Now we consider the alternative (ii). This implies $\nu(\alpha) = 1$ because a cubic polynomial with real coefficients must have at least one real root. We have $V \equiv V_1$, hence $n_2 = n_3 = 0$. As a corollary, we have $\phi(\bar{x}) = t_1 x^2$, $x \in \mathbb{R}^{n_1}$, where t_1 is the unique real root of $\chi_\alpha(t)$. In this case, $\psi \equiv \psi_3$ and (2.14) reduces to a single equation $\psi_3 R_3 = 0$. If $\psi_3 \neq 0$ then in view of (2.13) we have

$R_3(t_1) \equiv \frac{9\alpha}{2} + 12t_1^2 - 3(3 + 2\alpha)t_1 = 0$. Therefore $R_2(t)$ and $\chi_\alpha(t)$ have a common root $t = t_1$, which implies that their resultant must be zero:

$$\mathcal{R}(g, \chi_\alpha) \equiv -486\alpha(16\alpha + 3)(\alpha - 6)(\alpha + 3) = 0.$$

In the cases $\alpha = -3$ and $\alpha = 6$ the characteristic equation $\chi_\alpha(t) = 0$ has three real roots. Thus, $\alpha = -\frac{16}{3}$, in which case the unique real root is $t_1 = \frac{3}{4}$. But by virtue of (2.12), $n_1 = \frac{\alpha}{t_1} = -\frac{1}{4}$, a contradiction.

It remains to consider $\psi_3 \equiv 0$. In this case we have $\psi \equiv 0$ and from (2.7) we obtain $9\alpha\bar{x}^2 + \phi_{\bar{x}}^2 - \alpha\phi = 0$, which is possible only if $\alpha \in \{0, -3, -\frac{1}{2}\}$. By the assumption, $\nu(\alpha) = 1$, hence $\alpha = -\frac{1}{2}$ and the unique real root in this case is $t_1 = 1$. A contradiction follows because $n_1 = \frac{\alpha}{t_1} = -\frac{1}{2}$, so lemma is proved completely. \square

Now we are ready to give a proof of the main result of this section.

PROOF OF THEOREM 1. By virtue of (2.1),

$$\Delta f = 2x_n(\alpha + 3), \quad (2.17)$$

hence it suffices to show that for any radial eigencubic given by (2.1) and normalized by $c = 1$, there holds $\alpha = -3$. Let f be an arbitrary such eigencubic. By Lemma 2.2 we have $\nu(\alpha) = 3$, i.e. the characteristic polynomial (2.10) has three distinct real roots $t_1 < t_2 < t_3$. Since by (2.10) $t_1 t_2 t_3 = -9\alpha \neq 0$, we have $t_i \neq 0$.

Now we proceed by contradiction and suppose that $\alpha \neq -3$. Then $\psi \not\equiv 0$ because otherwise (2.8) would imply three alternatives $\alpha \in \{0, -3, -\frac{1}{2}\}$, of which only $\alpha = -3$ yields $\nu(\alpha) = 3$, a contradiction. Let $\psi = \sum_{|q|=3} \psi_q$ be the decomposition of ψ into homogeneous parts $\psi_q \in V^{*\otimes q}$. We claim that there exists $q \neq (1, 1, 1)$ such that $\psi_q \not\equiv 0$. Indeed, let us suppose the contrary, i.e. that $\psi \equiv \psi_{111}$. Since $\psi \not\equiv 0$, we have $n_k = \dim V_k > 0$ for $k = 1, 2, 3$. Furthermore, $\psi \in V_1^* \otimes V_2^* \otimes V_3^*$ implies

$$\psi_{\bar{x}}^2 \equiv \sum_{i=1}^{n-1} \psi_{x_i}^2 \in \sum_{j \neq k} V_j^{*\otimes 2} \otimes V_k^{*\otimes 2} =: W.$$

We also have from the diagonal form of ϕ

$$\psi_{\bar{x}}^\top \phi_{\bar{x}\bar{x}} \psi_{\bar{x}} = 2 \sum_{i=1}^{n-1} \phi_i \psi_{x_i}^2 \in W.$$

Similarly, $\phi_{\bar{x}}^\top \psi_{\bar{x}\bar{x}} \psi_{\bar{x}} \in W$ because if $x_i \in V_k^*$ and $x_j \in V_l^*$ for $k \neq l$ then

$$\phi_{x_i} \psi_{x_i x_j} \psi_{x_j} \in V_k^* \otimes V_m^* \otimes (V_k^* \otimes V_m^*) \subset W,$$

where $\{k, l, m\} = \{1, 2, 3\}$, and if $k = l$ then $\psi = \psi_{111}$ yields $\psi_{x_i x_j} = 0$. This shows that the left hand side of (2.7) belongs to W .

On the other hand, combining terms in the right hand side of (2.7), we get

$$\begin{aligned} 2 \sum_{i=1}^{n-1} \phi_i x_i^2 \cdot \sum_{i=1}^{n-1} (4\phi_i^2 - \alpha\phi_i + 9\alpha)x_i^2 &= 2 \sum_{j=1}^3 t_j u_j^2 \cdot \sum_{j=1}^3 (4t_j^2 - \alpha t_j + 9\alpha)u_j^2 \\ &= 2 \sum_{j=1}^3 t_j (4t_j^2 - \alpha t_j + 9\alpha)u_j^4 + h, \end{aligned}$$

where $h \in W$ and u_i is the projection of x onto V_i . This yields $\sum_{j=1}^3 c_i u_i^4 \in W$, where $c_i = t_j(4t_j^2 - \alpha t_j + 9\alpha)$, hence $c_1 = c_2 = c_3 = 0$. Since $t_j \neq 0$, we conclude that $4t_j^2 - \alpha t_j + 9\alpha = 0$ for all $j = 1, 2, 3$. But this yields that the quadratic polynomial $4t^2 - \alpha t + 9\alpha$ has three distinct

real roots, a contradiction. This proves that there exists $q \neq (1, 1, 1)$ such that $\psi_q \neq 0$. Applying Lemma 2.1, we see that the corresponding R -coefficient must be zero. This gives

$$0 = \prod_{q \neq (1,1,1)} R_q. \quad (2.18)$$

Write the latter product as $\rho_1 \rho_2$, where

$$\rho_1 = R_{300} R_{030} R_{003}, \quad \rho_2 = R_{210} R_{201} R_{120} R_{021} R_{102} R_{012}.$$

Then ρ_1 and ρ_2 are symmetric functions of t_i , $i = 1, 2, 3$, hence can be expressed as polynomials in α . For instance, in order to find ρ_2 we note that

$$\begin{aligned} R_{210} &= 2t_2^2 + 6t_1^2 + 4t_1 t_2 - (3 + 2\alpha)(2t_1 + t_2) + \frac{9\alpha}{2}, \\ R_{201} &= 2t_3^2 + 6t_1^2 + 4t_1 t_3 - (3 + 2\alpha)(2t_1 + t_3) + \frac{9\alpha}{2}, \end{aligned}$$

hence eliminating t_2 and t_3 in $R_{210} R_{201}$ by virtue of Viète's formulas, we get

$$R_{210} R_{201} = 4(2t_1 - 3)^2 (12t_1^2 - 8\alpha t_1 + 3\alpha) \equiv 16(t_1 - \frac{3}{2})^2 \chi'_\alpha(t_1),$$

which yields

$$\rho_2 = 16^3 \prod_{i=1}^3 (t_i - \frac{3}{2})^2 \prod_{i=1}^3 \chi'_\alpha(t_i) = -4^3 \chi_\alpha(3/2)^2 \mathcal{D}(\chi_\alpha) \equiv -2^4 3^4 (\alpha + 3)^2 \mathcal{D}(\chi_\alpha),$$

where $\mathcal{D}(\chi_\alpha) = -4\chi'(\alpha, t_1)\chi'(\alpha, t_2)\chi'(\alpha, t_3)$ is the discriminant of χ_α . By our assumption, the characteristic polynomial χ_α has exactly three distinct real roots, hence $\mathcal{D}(\chi_\alpha) \neq 0$. Thus, in view of $\alpha \neq -3$ we conclude that $\rho_2 \neq 0$. This yields by virtue of Lemma 2.1, that $\phi_q \equiv 0$ for any q obtained from $(1, 2, 0)$ by permutations. In particular,

$$\psi = \psi_{111} + \psi_{300} + \psi_{030} + \psi_{003}. \quad (2.19)$$

On the other hand, $\rho_2 \neq 0$ yields by virtue of (2.18) that $\rho_1 = 0$. We have from (2.13)

$$\begin{aligned} 0 = \rho_1 &= \prod_{k=1}^3 (\frac{9\alpha}{2} + 12t_k^2 - 3(3 + 2\alpha)t_k) = 12^3 \prod_{k=1}^3 (t_k - \frac{3}{4})(t_k - \frac{\alpha}{2}), \\ &= \frac{12^3}{4^6} \cdot \chi_\alpha(3/4) \chi_\alpha(\alpha/2) = -\frac{3^5}{2^{11}} \cdot \alpha(\alpha + 3)(\alpha - 6)(16\alpha + 3). \end{aligned}$$

It is easily verified that, except for $\alpha = -3$, only for $\alpha = 6$ the characteristic polynomial has three real roots. By solving the corresponding characteristic equation $\chi_6(t) \equiv 2(t - 3)(2t^2 - 6t - 9) = 0$, we obtain $t_1 = 3$ and $t_{2,3} = \frac{3 \pm 3\sqrt{3}}{2}$. Then (2.12) yields the relation between the dimensions $n_i = \dim V_i$ of the corresponding eigen spaces V_i :

$$\frac{6n_1 + 3n_2 + 3n_3}{2} + \frac{\sqrt{3}}{2}(n_2 - n_3) = 6,$$

Since n_i are nonnegative integers, we find $n_2 = n_3$ and $n_1 + n_2 = 2$, which gives the following admissible triples

$$(n_1, n_2, n_3) \in \{(2, 0, 0), (1, 1, 1), (0, 2, 2)\}.$$

Substituting the found t_i into (2.13), we obtain additionally that R_{030} and R_{003} are non-zero, which by Lemma 2.1 and (2.19) yields $\psi = \psi_{111} + \psi_{300}$. Note that ψ_{111} is harmonic because it is linear in each variable and ψ is harmonic by virtue of (2.4). Thus, ψ_{300} is harmonic. On the other hand, $\psi_{300} \neq 0$ because $\psi \neq \psi_{111}$. This yields $n_1 \geq 2$. This strikes the triples $(1, 1, 1)$ and $(0, 2, 2)$ from the list.

Consider the only remaining triple $(n_1, n_2, n_3) = (2, 0, 0)$. In this case V_2 and V_3 are trivial, and V_1 is two-dimensional, hence $\psi \equiv \psi_{300}$. Since $t_1 = 3$, we have $\phi = 3(x_1^2 + x_2^2)$. This implies for the left hand side of (2.7)

$$2(3 + \alpha)\psi_{\bar{x}}^2 - \psi_{\bar{x}}^\top \phi_{\bar{x}\bar{x}} \psi_{\bar{x}} - 2\phi_{\bar{x}}^\top \psi_{\bar{x}\bar{x}} \psi_{\bar{x}} 18\psi_{\bar{x}}^2 - 6\psi_{\bar{x}}^2 - 24\psi_{\bar{x}}^2 = -12\psi_{\bar{x}}^2.$$

On the other hand, the right hand side is strictly positive:

$$2\phi(9\alpha\bar{x}^2 + \phi_{\bar{x}}^2 - \alpha\phi) = 2^4 \cdot 3^3 (x_1^2 + x_2^2)^2.$$

The contradiction shows that $\alpha \neq 6$, the theorem is proved completely. \square

3. A hidden Clifford structure

Let us consider an arbitrary radial eigencubic f given in the normal form (2.1) normalized by $c = 1$. Then by Theorem 1, any radial eigencubic is harmonic, hence (2.17) yields $\alpha = -3$ in (2.2) (equivalently, $\lambda(f) = -54$ in (1.4)) and we have for the characteristic polynomial (2.10)

$$\chi_{-3}(t) = 4t^3 + 12t^2 - 9t - 27 = 4(t+3)(t - \frac{3}{2})(t + \frac{3}{2}),$$

which yields $t_1 = -3$, $t_2 = -\frac{3}{2}$ and $t_3 = \frac{3}{2}$. Write $\mathbb{R}^n = \text{span}(e_n) \oplus V$ and denote by $V = V_1 \oplus V_2 \oplus V_3$ the eigen decomposition of V associated with ϕ . Then

$$\phi(\bar{x}) = -3\xi^2 - \frac{3}{2}\eta^2 + \frac{3}{2}\zeta^2, \quad \text{where } \bar{x} = \xi \oplus \eta \oplus \zeta \in V \quad (3.1)$$

If $n_i = \dim V_i$ then in virtue of (2.12)

$$n_3 = 2n_1 + n_2 - 2, \quad n = 3n_1 + 2n_2 - 1. \quad (3.2)$$

A close examination of (2.13) shows that $R_q = 0$ vanish only if q is one of the following: (111), (102), (012), (030). This yields by virtue of Lemma 2.1

$$\psi = \psi_{111} + \psi_{102} + \psi_{012} + \psi_{030}, \quad \psi_q \in V^{*\otimes q}, \quad (3.3)$$

and

$$\Delta\psi_{102} = 0, \quad \Delta\psi_{012} = -\Delta\psi_{030} \quad (3.4)$$

by virtue of (2.4). The remaining equations (2.7) and (2.8) are read as follows:

$$\psi_{\bar{x}}^\top \phi_{\bar{x}\bar{x}} \psi_{\bar{x}} + 2\phi_{\bar{x}}^\top \psi_{\bar{x}\bar{x}} \psi_{\bar{x}} = \frac{27}{2}(\zeta^2 - 2\xi^2 - \eta^2)(5\eta^2 + 3\zeta^2), \quad (3.5)$$

$$2\phi \phi_{\bar{x}}^\top \psi_{\bar{x}} + \psi_{\bar{x}}^\top \psi_{\bar{x}\bar{x}} \psi_{\bar{x}} = 54(\xi^2 + \eta^2 + \zeta^2)\psi. \quad (3.6)$$

In summary, we have

Proposition 3.1. *Given a radial eigencubic f in \mathbb{R}^n , there is an orthogonal decomposition $\mathbb{R}^n = \text{Span}[e_n] \oplus V_1 \oplus V_2 \oplus V_3$, such that*

$$f = x_n^3 - \frac{3}{2}x_n(2\xi^2 + \eta^2 - \zeta^2) + \psi_{111} + \psi_{102} + \psi_{012} + \psi_{030}, \quad (3.7)$$

where $x = (\xi, \eta, \zeta, x_n)$, $\xi \in V_1$, $\eta \in V_2$, $\zeta \in V_3$, and $\dim V_i = n_i$ satisfy (3.2). Moreover, ψ satisfy (3.3) and (3.5)–(3.6). Conversely, if the cubic polynomial (3.7) satisfies (3.3) and (3.5)–(3.6) then f is a radial eigencubic.

Definition. Suppose a radial eigencubic f admits the normal form (3.7). Then the pair (n_1, n_2) is called the *type* of the normal form, where $\dim V_1 = n_1$ and $\dim V_2 = n_2$.

Proposition 3.2. *Let f be a radial eigencubic given in the normal form (3.7). Then*

$$3(\psi_{111})_\eta^2 + (\psi_{102})_\zeta^2 = 27\zeta^2\xi^2, \quad (3.8)$$

$$(\psi_{111})_\zeta^2 = 9\eta^2\xi^2, \quad (3.9)$$

$$(\psi_{111})_\zeta^\top (\psi_{102})_\zeta = 0, \quad (3.10)$$

$$6(\psi_{012})_\eta^2 + 4(\psi_{102})_\xi^2 = 27\zeta^4, \quad (3.11)$$

$$(\psi_{030})_\eta^2 = \frac{9}{2}\eta^4, \quad (3.12)$$

$$2(\psi_{111})_\eta^\top (\psi_{030})_\eta + (\psi_{111})_\zeta^\top (\psi_{012})_\zeta = 0, \quad (3.13)$$

$$2(\psi_{111})_\xi^2 + 2(\psi_{030})_\eta^\top (\psi_{012})_\eta - (\psi_{012})_\zeta^2 = -9\zeta^2\eta^2. \quad (3.14)$$

PROOF. We consider (3.5) as an identity in $V^{*\otimes q}$, $|q| = 4$. Let π_q denote the projection of $V^{*\otimes 4}$ onto $V^{*\otimes q}$ and let $S = \frac{27}{2}(\zeta^2 - 2\xi^2 - \eta^2)(5\eta^2 + 3\zeta^2)$ denote the right hand side of (3.5). Since $\psi_{\bar{x}}^\top \phi_{\bar{x}\bar{x}} \psi_{\bar{x}} = -6\psi_\xi^2 - 3\psi_\eta^2 + 3\psi_\zeta^2$ and

$$2\phi_{\bar{x}}^\top \psi_{\bar{x}\bar{x}} \psi_{\bar{x}} (\psi_{\bar{x}}^2)_{\bar{x}}^\top \phi_{\bar{x}} = \sum_{|q|=4} (-6q_1 - 3q_2 + 3q_3) \pi_q(\psi_{\bar{x}}^2),$$

we obtain from (3.5)

$$\begin{aligned} \pi_q(S) &= \pi_q(\psi_{\bar{x}}^\top \phi_{\bar{x}\bar{x}} \psi_{\bar{x}} + 2\phi_{\bar{x}}^\top \psi_{\bar{x}\bar{x}} \psi_{\bar{x}}) \\ &= \pi_q(-6\psi_\xi^2 - 3\psi_\eta^2 + 3\psi_\zeta^2) - 3(2q_1 + q_2 - q_3) \pi_q(\psi_{\bar{x}}^2) \\ &= -3(\beta_q + 2) \pi_q(\psi_\xi^2) - 3(\beta_q + 1) \pi_q(\psi_\eta^2) - 3(\beta_q - 1) \pi_q(\psi_\zeta^2), \end{aligned} \quad (3.15)$$

where $\beta_q = 2q_1 + q_2 - q_3$.

We have for $q = (202)$: $\beta_{202} = 2$ and $\pi_{202}(\psi_\xi^2) = 0$, whereas $\pi_{202}(\psi_\eta^2) = (\psi_{111})_\eta^2$ and $\pi_{202}(\psi_\zeta^2) = (\psi_{102})_\zeta^2$. This yields by (3.15)

$$\begin{aligned} \pi_{202}(S) &= -3(\beta_{202} + 1)(\psi_{111})_\eta^2 - 3(\beta_{202} - 1)(\psi_{102})_\zeta^2 \\ &= -9(\psi_{111})_\eta^2 - 3(\psi_{102})_\zeta^2, \end{aligned}$$

which proves (3.8) because $\pi_{202}(S) = -81\xi^2\zeta^2$.

Arguing similarly for $q = (220)$ one obtains

$$-135\eta^2\xi^2 \equiv \pi_{220}(S) = -3(\beta_{220} - 1)(\psi_{111})_\zeta^2 = -15(\psi_{111})_\zeta^2,$$

hence (3.9) follows. The remaining identities are established similarly. \square

Let us rewrite ψ_{111} and ψ_{102} in matrix form as follows:

$$\psi_{111} = 3\eta^\top P_\xi \zeta, \quad \psi_{102} = \frac{3\sqrt{3}}{2} \zeta^\top Q_\xi \zeta, \quad (3.16)$$

where

$$P_\xi = \sum_{i=1}^{n_1} \xi_i P_i, \quad Q_\xi = \sum_{i=1}^{n_1} \xi_i Q_i, \quad P_i \in \mathbb{R}^{n_2 \times n_3}, \quad Q_i \in \mathbb{R}_{\text{symm}}^{n_3 \times n_3}.$$

Here and in what follows, by $\mathbb{R}^{k \times m}$ we denote the vector space of matrices of the corresponding size and by $\mathbb{R}_{\text{symm}}^{m \times m}$ denote the space of symmetric matrices of size m . In addition to (3.16) it is convenient also to introduce the following matrix notation:

$$\psi_{111} = 3\eta^\top P_\eta \zeta \equiv 3\xi^\top N_\eta \zeta, \quad \psi_{012} = \frac{3\sqrt{2}}{2} \zeta^\top R_\eta \zeta, \quad (3.17)$$

and

$$H(\eta) = \frac{\sqrt{2}}{3} \nabla_\eta \psi_{030}(\eta), \quad (3.18)$$

where $N_\eta \in \mathbb{R}^{n_1 \times n_3}$, $R_\xi \in \mathbb{R}_{\text{sym}}^{n_3 \times n_3}$. Then the equations (3.8)–(3.14) are rewritten in matrix notation as follows:

$$\sum_{i=1}^{n_2} (\xi^\top N_i \zeta)^2 + \zeta^\top Q_\xi^2 \zeta = \zeta^2 \xi^2 \quad (3.19)$$

$$N_\eta N_\eta^\top = \eta^2 \mathbf{1}_{V_1}, \quad (3.20)$$

$$Q_\xi N_\eta^\top \xi = 0, \quad (3.21)$$

$$\sum_{i=1}^{n_1} (\zeta^\top Q_i \zeta)^2 + \sum_{j=1}^{n_2} (\zeta^\top R_j \zeta)^2 = \zeta^4, \quad (3.22)$$

$$H^\top H = \eta^4, \quad (3.23)$$

$$N_H + N_\eta R_\eta = 0, \quad (3.24)$$

$$2N_\eta^\top N_\eta + R_H - 2R_\eta^2 + \eta^2 \mathbf{1}_{V_3} = 0. \quad (3.25)$$

Proposition 3.3 (Hidden Clifford structure). *Let f be a radial eigencubic with the normal form (3.7) of type (n_1, n_2) . Then the cubic form*

$$C(z) := \frac{2}{3}(\psi_{111} + \frac{1}{\sqrt{3}}\psi_{102}), \quad z = (\eta, \zeta) \in \mathbb{R}^{2n_1+2n_2-2},$$

is an eigencubic of Clifford type. In particular,

$$n_1 - 1 \leq \rho(n_2 + n_1 - 1), \quad (3.26)$$

where ρ is the Hurwitz-Radon function (1.7).

PROOF. The case $n_1 = 0$ is trivial. Let us suppose that f be a radial eigencubic with the normal form (3.7) of type (n_1, n_2) , where $n_1 = \dim V_1 \geq 1$. By using (3.16), $(\psi_{111})_\eta = 3P_\xi \zeta$, $(\psi_{111})_\zeta = 3P_\xi^\top \eta$, and $(\psi_{102})_\zeta = 6\sqrt{3}Q_\xi \zeta$, so that (3.8), (3.9) and (3.10) become the following matrix identities:

$$P_\xi^\top P_\xi + Q_\xi^2 = \xi^2 \mathbf{1}_{V_3}, \quad P_\xi P_\xi^\top = \xi^2 \mathbf{1}_{V_2}, \quad P_\xi Q_\xi = 0. \quad (3.27)$$

The latter is equivalent to that the symmetric matrices

$$E_i = \begin{pmatrix} 0 & P_i \\ P_i^\top & Q_i \end{pmatrix} \quad (3.28)$$

satisfy

$$E_i E_j + E_j E_i = 2\delta_{ij} \mathbf{1}_{V_2 \oplus V_3} \quad 1 \leq i, j \leq n_1,$$

which implies that $\{E_i\}_{1 \leq i \leq n_1}$ is a symmetric Clifford system in $V_2 \oplus V_3 = \mathbb{R}^{n_2+n_3} = \mathbb{R}^{2(n_1+n_2-1)}$. In particular, this yields that $C(z) = \frac{2}{3}(\psi_{111} + \frac{1}{\sqrt{3}}\psi_{102})$ is indeed a Clifford eigencubic, cf. (1.9). By using (1.6) we get (3.26). \square

The above results naturally yields a classification of radial eigencubics with $n_2 = 0$.

Proposition 3.4. *A radial eigencubic f admits the normal form having the property $n_2 = 0$ if and only if f is congruent to either the Cartan polynomial θ_ℓ , $\ell \in \{1, 2, 4, 8\}$ or $\theta_0 := x_2^3 - 3x_2x_1^2$. Furthermore, in that case $n_1 = \ell + 1$.*

PROOF. First remark by virtue of (1.11) that for the Cartan polynomials θ_ℓ , $\ell = 0, 1, 2, 4, 8$, there holds $n_1 = \ell + 1$ and $n_2 = 0$.

Now suppose that f is an arbitrary radial eigencubic which admits the normal form (3.7) with the property that $n_2 = 0$. Then $V_2 = \{0\}$ and (3.3) yields $\psi \equiv \psi_{102}$. Moreover, (3.2)

yields $n_3 = 2n_1 - 2$. If $n_1 = 1$ then $n_3 = 0$ and $\psi_{102} \equiv 0$, so that (3.7) implies the trivial case, $f = x_2^3 - 3\xi_1^2 x_2 \equiv \theta_0$. Hence we can assume that $n_1 \geq 2$. Then $n_3 = \dim V_3 \geq 1$ and

$$f = x_n^3 - \frac{3}{2}x_n(2\xi^2 - \zeta^2) + \psi_{102}.$$

In particular, by (3.11) and (3.8) $\psi_x^2 \equiv (\psi_{012})_\xi^2 + (\psi_{102})_\zeta^2 = \frac{27}{4}(\zeta^2 + 4\xi^2)\zeta^2$, which implies

$$|\nabla f|^2 = 9(x_n^2 + \xi^2 + \zeta^2)^2 \equiv 9x^4.$$

Taking into account that f is harmonic, we conclude that f satisfies the Muntzer-Cartan equations (1.10), thus by the Cartan theorem f must be congruent to θ_ℓ for some $\ell \in \{1, 2, 4, 8\}$. This yields $3\ell + 2 = n \equiv n_1 + n_3 + 1 = 3n_1 - 1$, hence $n_1 = \ell + 1$ as required. \square

4. Proof of Theorem 3

We split the proof of Theorem 3 into two steps: the ‘if’-part will be established in Proposition 4.1 below, and the ‘only if’-part will be given in Corollary 5.3.

Proposition 4.1. *Let f be a radial eigencubic in the normal form (3.7). If ψ_{030} is reducible and not identically zero then f is of Clifford type.*

PROOF OF PROPOSITION 4.1. By the assumption, ψ_{030} is reducible and not identically zero, hence by the Eiconal Cubic Theorem there exist orthogonal coordinates $(\eta_1, \dots, \eta_{n_2})$ in $V_2 = \mathbb{R}^{n_2}$ such that

$$\psi_{030} = \frac{1}{\sqrt{2}}(\eta_{n_2}^3 - 3\eta_{n_2}\bar{\eta}^2), \quad \bar{\eta} = (\eta_1, \dots, \eta_{n_2-1}) \in V_2'.$$

Then we have for the vector field (3.18): $H = (\eta_{n_2}^2 - \bar{\eta}^2, -2\eta_{n_2}\bar{\eta})$. This yields

$$N_H = -2\eta_{n_2}N_{\bar{\eta}} + (\eta_{n_2}^2 - \bar{\eta}^2)N_{n_2}, \quad R_H = -2\eta_{n_2}R_{\bar{\eta}} + (\eta_{n_2}^2 - \bar{\eta}^2)R_{n_2},$$

where $N_\eta \in \mathbb{R}^{n_1 \times n_3}$ and $R_\eta \in \mathbb{R}_{\text{symm}}^{n_3 \times n_3}$ are defined by (3.17). Thus (3.24) and (3.25) becomes respectively

$$-2\eta_{n_2}N_{\bar{\eta}} + (\eta_{n_2}^2 - \bar{\eta}^2)N_{n_2} + N_\eta R_\eta = 0 \tag{4.1}$$

and

$$2N_\eta^\top N_\eta - 2\eta_{n_2}R_{\bar{\eta}} + (\eta_{n_2}^2 - \bar{\eta}^2)R_{n_2} - 2R_\eta^2 - \eta^2 \mathbf{1}_{V_3} = 0. \tag{4.2}$$

By identifying the coefficients of $\eta_{n_2}^2$ in the latter relations one finds

$$\begin{aligned} N_{n_2}(\mathbf{1}_{V_3} + R_{n_2}) &= 0, \\ 2N_{n_2}^\top N_{n_2} &= 2R_{n_2}^2 - R_{n_2} - \mathbf{1}_{V_3}, \end{aligned} \tag{4.3}$$

which yields $(2R_{n_2}^2 - R_{n_2} - \mathbf{1}_{V_3})(\mathbf{1}_{V_3} + R_{n_2}) = 0$. The latter equation shows that R_{n_2} has eigenvalues ± 1 and $-\frac{1}{2}$. Let $V_3 = Y \oplus Z \oplus W$ be the corresponding eigen decomposition of R_{n_2} and let $\zeta = (y, z, w)$ denote the associated decomposition of a typical vector $\zeta \in V_3$. We have

$$\dim Y + \dim Z + \dim W = n_3 \equiv 2n_1 + n_2 - 2 \tag{4.4}$$

and

$$\text{tr } R_{n_2} = \dim Y - \dim Z - \frac{1}{2} \dim W. \tag{4.5}$$

On the other hand, (3.20) yields

$$N_{n_2}N_{n_2}^\top = \mathbf{1}_{V_1}, \tag{4.6}$$

hence we find from the second equation in (4.3)

$$2n_1 = 2 \text{tr } N_{n_2}N_{n_2}^\top = 2 \text{tr } N_{n_2}^\top N_{n_2} = \text{tr}(2R_{n_2}^2 - R_{n_2} - \mathbf{1}_{V_3}) = 2 \dim Z.$$

Thus $\dim Z = n_1$. Also, in view of the second relation in (3.4),

$$\text{tr } R_\eta = \frac{1}{3\sqrt{2}} \Delta_\zeta \psi_{012} = -\frac{1}{3\sqrt{2}} \Delta_\eta \psi_{003} = (n_2 - 2)\eta_{n_2},$$

hence $\text{tr } R_{n_2} = n_2 - 2$ and $\text{tr } R_i = 0$ for $1 \leq i \leq n_2 - 1$. Combining this with (4.4) and (4.5) we obtain

$$\dim W = 0, \quad \dim Y = n_1 + n_2 - 2, \quad \dim Z = n_1.$$

In particular, $V_3 = Y \oplus Z$ and we can write R_{n_2} in the block form

$$R_{n_2} = \begin{pmatrix} \mathbf{1}_Y & 0 \\ 0 & -\mathbf{1}_Z \end{pmatrix}, \quad (4.7)$$

which yields in view of (4.3)

$$N_{n_2}^\top N_{n_2} = 2R_{n_2}^2 - R_{n_2} - \mathbf{1}_{V_3} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_Z \end{pmatrix}. \quad (4.8)$$

Regarding N_{n_2} as a mapping from $V_3 = Y \oplus Z$ into V_1 , we obtain from (4.8) the corresponding block representation: $N_{n_2} = (\mathbf{0}, U)$, where $U : Z \rightarrow V_1$ satisfies, by virtue of (4.8), $U^\top U = \mathbf{1}_Z$ and also, in view of (4.6), $UU^\top = \mathbf{1}_{V_1}$, hence U is an isometry. Thus, we may assume without loss of generality that the orthogonal coordinates in V_1 and Z are agreed so that $U = \mathbf{1}$ is the unit matrix (of size $\dim V_1 = \dim Z = n_1$). This yields $N_{n_2} = (\mathbf{0}, \mathbf{1})$.

On substituting (4.7) into (3.22), we obtain

$$\sum_{i=1}^{n_1} (\zeta^\top Q_i \zeta)^2 + \sum_{j=1}^{n_2-1} (\zeta^\top R_j \zeta)^2 = (y^2 + z^2)^2 - (y^2 - z^2)^2 = 4y^2 z^2,$$

where

$$\zeta = (y, z) \in Y \oplus Z, \quad (4.9)$$

which implies

$$Q_j = \begin{pmatrix} 0 & \alpha_j \\ \alpha_j^\top & 0 \end{pmatrix}, \quad R_i = \begin{pmatrix} 0 & \beta_i \\ \beta_i^\top & 0 \end{pmatrix}, \quad 1 \leq i \leq n_2 - 1. \quad (4.10)$$

Coming back to (3.20), it can be seen that

$$N_{n_2} N_{\bar{\eta}}^\top + N_{\bar{\eta}} N_{n_2}^\top = 0 \quad \text{and} \quad N_{\bar{\eta}} N_{\bar{\eta}}^\top = \bar{\eta}^2 \mathbf{1}_{V_1}. \quad (4.11)$$

Regarding $N_{\bar{\eta}} \equiv \sum_{i=1}^{n_2-1} N_i \eta_i$ as a mapping from $V_3 = Y \oplus Z$ into V_1 and rewriting it in the block form as $(a_{\bar{\eta}}, b_{\bar{\eta}})$, we deduce from (4.11) that

$$b_{\bar{\eta}} + b_{\bar{\eta}}^\top = 0, \quad a_{\bar{\eta}} a_{\bar{\eta}}^\top + b_{\bar{\eta}} b_{\bar{\eta}}^\top = \bar{\eta}^2 \mathbf{1}_{V_1}.$$

Also, by identifying the coefficients of η_{n_2} in (4.1), one finds $2N_{\bar{\eta}} = N_{n_2} R_{\bar{\eta}} + N_{\bar{\eta}} R_{n_2}$, which yields $a_{\bar{\eta}} = \beta_{\bar{\eta}}^\top$ and $b_{\bar{\eta}} = 0$. Thus,

$$N_{\bar{\eta}} = (\beta_{\bar{\eta}}^\top, \eta_{n_2} \mathbf{1}) \quad (4.12)$$

Now we consider the normal form (3.7) and rewrite it in the Clifford form (??). To this end, let us first introduce the new orthogonal coordinates

$$x_n = \frac{x_0 + \sqrt{2}y_0}{\sqrt{3}}, \quad \eta_{n_2} = \frac{\sqrt{2}x_0 - y_0}{\sqrt{3}}$$

in the (x_n, η_{n_2}) -plane. Then the following identity is verified by a straightforward calculation:

$$x_n^3 + \frac{3}{2}x_n(-\eta_{n_2}^2 - \bar{\eta}^2 + y^2) + \frac{\sqrt{2}}{2}(\eta_{n_2}^3 - 3\eta_{n_2}\bar{\eta}^2) + \frac{3\sqrt{2}}{2}\eta_{n_2}y^2 = \frac{3\sqrt{3}}{2}x_0(y_0^2 + y^2 - \bar{\eta}^2).$$

Taking into account (4.9), (4.12) and (4.10), we rewrite (3.7) in the new coordinates as follows

$$f = \frac{3\sqrt{3}}{2}x_0(y_0^2 + y^2 - \bar{\eta}^2) + \frac{3x_n}{2}(z^2 - 2\xi^2) - \frac{3\sqrt{3}}{2}\eta_{n_2}z^2 + 3\eta_{n_2}\xi^\top z + \Omega, \quad (4.13)$$

where

$$\Omega = 3y^\top \beta_{\bar{\eta}}(\xi + z\sqrt{2}) + 3\sqrt{3}y^\top \alpha_\xi z.$$

Let us introduce the new orthogonal coordinates in $V_1 \oplus Z$ by virtue of

$$s = \frac{\xi + \sqrt{2}z}{\sqrt{3}}, \quad t = \frac{z - \sqrt{2}\xi}{\sqrt{3}}.$$

Then

$$\frac{3x_n}{2}(z^2 - 2\xi^2) - \frac{3\sqrt{3}}{2}\eta_{n_2}z^2 + 3\eta_{n_2}\xi^\top z = -\frac{3\sqrt{3}}{2}(x_0t^2 - 2y_0s^\top t),$$

and (4.13) becomes

$$f_1 = x_0(y_0^2 + y^2 - \bar{\eta}^2 - t^2) + 2y_0s^\top t + \Omega, \quad (4.14)$$

where $f_1 := \frac{2\sqrt{3}}{9}f$ and

$$\Omega = 2y^\top \beta_{\bar{\eta}}s + 2y^\top \alpha_\xi z. \quad (4.15)$$

Next, note that in view of (4.10) and (3.21)

$$\alpha_\xi \xi = 0, \quad (4.16)$$

hence setting $\xi = \xi' + \xi''$ in (4.16) we get $\alpha_{\xi'}\xi'' = -\alpha_{\xi''}\xi'$, which yields for the last term in (4.15)

$$y^\top \alpha_\xi z = \frac{1}{3}y^\top \alpha_{s-\sqrt{2}t}(t + \sqrt{2}s) = y^\top \alpha_s t.$$

Thus $\Omega = 2y^\top \beta_{\bar{\eta}}s + 2y^\top \alpha_s t$, hence we arrive at the following expression

$$f_1 = x_0(y_0^2 + y^2 - \bar{\eta}^2 - t^2) + 2y_0s^\top t + 2y^\top \beta_{\bar{\eta}}s + 2y^\top \alpha_s t. \quad (4.17)$$

In order to show that the last expression is indeed a Clifford representation, we rewrite f_1 in matrix notation as follows. We combine the coordinates as follows:

$$\tilde{y} = (y_0, y) \in \tilde{Y}, \quad \tilde{z} = (\bar{\eta}, t) \in \tilde{Z},$$

where $Y \cong Z \cong \mathbb{R}^{n_1+n_2-1}$, and set $U = \tilde{Y} \oplus \tilde{Z}$ and $u = (\tilde{y}, \tilde{z}) \in U$. Then (4.17) becomes

$$f_1 = x_0 u^\top A_0 u + \sum_{i=1}^{n_1} s_i u^\top A_i u \equiv x_0 u^\top A_0 u + u^\top A_s u \quad (4.18)$$

where the matrices

$$A_0 = \begin{pmatrix} \mathbf{1}_{\tilde{Y}} & 0 \\ 0 & -\mathbf{1}_{\tilde{Z}} \end{pmatrix}, \quad A_i = \begin{pmatrix} 0 & D_i \\ D_i^\top & 0 \end{pmatrix},$$

are written in the block form with respect to the orthogonal decomposition $U = \tilde{Y} \oplus \tilde{Z}$. Here the matrix $D_s = \sum_{i=1}^{n_1} s_i D_i$ is defined by virtue of

$$2y_0s^\top t + 2y^\top \beta_{\bar{\eta}}s + 2y^\top \alpha_s t = \tilde{y}^\top D_s \tilde{z},$$

in other words,

$$D_s = \begin{pmatrix} 0 & s^\top \\ G_s & \alpha_s \end{pmatrix},$$

where the latter block-form is associated with the vector decompositions $\tilde{y} = (y_0, y)$ and $\tilde{z} = (\bar{\eta}, t)$, and the matrix G_s is determined by dualizing

$$y^\top \beta_{\bar{\eta}}s = y^\top G_s \bar{\eta}. \quad (4.19)$$

Coming back to (4.18), we see that it suffices to show that $\{A_0, A_1, \dots, A_{n_1}\}$ is a symmetric Clifford system in $\mathbb{R}^{2(n_1+n_2-1)}$. This will be done, in view of the explicit form of A_0 and the equality $\dim \tilde{Y} = \dim \tilde{Z}$, if we show that the subsystem $\{A_1, \dots, A_{n_1}\}$ is also a symmetric Clifford system, which is, in its turn, is equivalent to the following two identities: $D_s D_s^\top = s^2 \mathbf{1}_{\tilde{Y}}$ and $D_s^\top D_s = s^2 \mathbf{1}_{\tilde{Z}}$. Since the matrix D_s is square, it suffices to prove, e.g., the first of the last two relations. To this end, we write by virtue of (4.16)

$$D_s D_s^\top = \begin{pmatrix} s^2 & s^\top \alpha_s^\top \\ \alpha_s s & G_s G_s^\top + \alpha_s \alpha_s^\top \end{pmatrix} = \begin{pmatrix} s^2 & 0 \\ 0 & G_s G_s^\top + \alpha_s \alpha_s^\top \end{pmatrix}.$$

From (4.19) we have

$$y^\top G_s G_s^\top y = |G_s^\top y|^2 = \sum_{i=1}^{n_2-1} (y^\top G_s e_i)^2 = \sum_{i=1}^{n_2-1} (y^\top \beta_i s)^2, \quad (4.20)$$

where $\{e_i\}$ is an orthonormal basis in $V'_2 = \{(\bar{\eta}, 0) \in V_2\}$. On the other hand, by rewriting (3.19) by virtue of (4.12), we obtain

$$\xi^\top z + \sum_{i=1}^{n_2-1} (y^\top \beta_i \xi)^2 + y^\top \alpha_\xi \alpha_\xi^\top y + z^\top \alpha_\xi^\top \alpha_\xi z = (y^2 + z^2) \xi^2,$$

and setting $z = 0$ and $\xi = s$ in the latter identity we get in view of (4.20)

$$y^\top G_s G_s^\top y + y^\top \alpha_s \alpha_s^\top y = y^2 s^2,$$

i.e. $G_s G_s^\top + \alpha_s \alpha_s^\top = s^2 \mathbf{1}_{\tilde{Y}}$. This yields $D_s D_s^\top = s^2 \mathbf{1}_{\tilde{Y}}$ as required. Thus f_1 is a Clifford eigencubic and the proposition is proved completely. \square

Proposition 4.2. *If f is an exceptional eigencubic in the normal form (3.7) then ψ_{030} is either reducible or identically zero. Furthermore for an exceptional eigencubic $n_2 \in \{0, 5, 8, 14, 26\}$ and the triple (n_1, n_2, n_3) can take only the values presented in Table 2.*

n_1	2	3	5	9	0	1	2	4	0	1	2	3	5	9	0	1	2	3	0	1	2	3	7
n_2	0	0	0	0	5	5	5	5	8	8	8	8	8	8	14	14	14	14	26	26	26	26	26
n_3	2	4	8	16	3	5	7	11	6	8	10	12	16	24	12	14	16	18	24	26	28	30	38
n	5	8	14	26	9	12	15	21	15	18	21	24	30	42	27	30	33	36	51	54	57	60	72

TABLE 2. Possible type of exceptional eigencubics

PROOF. Let f be an exceptional eigencubic given in the normal form (3.7). Then by Proposition 4.1, ψ_{030} is either irreducible or identically zero. This implies by virtue of the Eiconal Cubic Theorem that $n_2 \in \{0, 5, 8, 14, 26\}$. On the other hand, by Proposition 3.3 we have

$$n_1 - 1 \leq \rho(n_2 + n_1 - 1). \quad (4.21)$$

Note that the following elementary inequality for the Hurwitz-Radon function holds:

$$\rho(m) \leq \frac{1}{2}(m + 8), \quad m \geq 1. \quad (4.22)$$

Indeed, by the definition (1.7) $\rho(m) = 8a + 2^b$, where $m = 2^s n_1$ with n_1 an odd number and $s = 4a + b$, $b = 0, 1, 2, 3$. For these b , $2^b \leq 2b + 2$, hence $\rho(m) \leq 8a + 2b + 2$. On the other hand,

$$8a + 2b + 2 = 2s + 2 \leq 2^{s-1} + 4 \leq \frac{1}{2}(2^s n_1 + 8),$$

which proves (4.22).

By (4.21) and (4.22), $n_1 - 1 \leq 2n_2 + 16$. Thus, given $n_2 \in \{0, 5, 8, 14, 26\}$ it suffices to examine (4.21) only for the numbers of n_1 for which $n_1 \leq 2n_2 + 17$. This easily yields the numbers presented in Table 2. \square

5. Proof of Theorem 2

We shall establish the cubic trace formula for radial eigencubic of Clifford type and exceptional eigencubics, separately in Proposition 5.2 and Proposition 5.1 below. We start with some definitions and lemma.

Definition. We say that an arbitrary cubic polynomial f in \mathbb{R}^n possess the *quadratic trace identity* (with the constant $\beta \in \mathbb{R}$) if

$$\text{tr Hess}^2 f = \beta x^2, \quad \beta = \beta(f) \in \mathbb{R}. \quad (5.1)$$

Similarly, we say f possess the *cubic trace identity* if it satisfies

$$\text{tr Hess}^3 f = \alpha f, \quad (5.2)$$

for some $\alpha \in \mathbb{R}$.

We shall also make use the following quadratic form

$$\sigma_2(f) = -\frac{\text{tr Hess}^2(f)}{\lambda}, \quad \text{where } \lambda = \frac{L(f)}{x^2 f}, \quad (5.3)$$

which spectrum is a congruence invariant of f in view of the invariant properties of the operator L .

Lemma 5.1. *For any radial eigencubic f in \mathbb{R}^n the following holds.*

- (i) *If f possess the quadratic trace identity with β then f posses the cubic trace identity with the constant $\alpha = -(n+6)\lambda - 6\beta$.*
- (ii) *If f possess the cubic trace identity with α then any normal form of f has type (n_1, n_2) , where $n_1 = \frac{\alpha}{3\lambda} + 1$ and $n_2 = \frac{n-3n_1+1}{2}$.*

In particular, if a radial eigencubic f possesses the cubic trace identity then its type is uniquely determined by the congruence class of f .

PROOF. (i) Since f is harmonic, we find from (1.4) that $\sum_{i,j=1}^n f_{x_i} f_{x_i x_j} f_{x_j} = -\lambda x^2 f$, hence applying the Laplacian to the latter identity and using the homogeneity of f we obtain

$$2 \text{tr Hess}^3 f + 4 \sum_{i,j,k=1}^n f_{x_i x_j} f_{x_i x_j x_k} f_{x_k} = -\lambda(2n+12)f,$$

On the other hand, by our assumption $\text{tr Hess}^2 f = \beta x^2$, hence

$$\sum_{i,j,k=1}^n f_{x_i x_j} f_{x_i x_j x_k} f_{x_k} \equiv \frac{1}{2} \langle \nabla f, \nabla \text{tr Hess}^2 f \rangle = \beta \sum_{k=1}^n x_k f_{x_k} = 3\beta f,$$

which yields $\text{tr Hess}^3 f = -((n+6)\lambda + 6\beta)f$ and thereby proves the first claim of the proposition.

(ii) Now suppose that f possess the cubic trace identity (5.2). It follows from the invariant properties of the operator L , see for instance [H], that the ratio

$$f \rightarrow \frac{\alpha}{\lambda} \equiv \frac{\text{tr Hess}^3 f}{f \cdot \lambda} \in \mathbb{R} \quad (5.4)$$

is invariant under orthogonal substitutions and dilatations, hence it suffices to establish (ii) in assumption that f is given in the normal form, e.g. by virtue of (2.1) and normalized by $c = 1$. In that case, the Hessian matrix of f has the following block form associated with the orthogonal decomposition $\mathbb{R}^n = \text{span}(e_n) \oplus \text{span}(e_n)^\perp$:

$$\text{Hess} f = \begin{pmatrix} 6x_n & \phi_x^\top \\ \phi_x & x_n \phi_{xx} + \psi_{xx} \end{pmatrix},$$

hence

$$\begin{aligned}\mathrm{tr} \mathrm{Hess}^3 f &= 216x_n^3 + 18x_n\phi_x^2 + 3\mathrm{tr} \phi_x\phi_x^\top (x_n\phi_{xx} + \psi_{xx}) + \mathrm{tr}(x_n\phi_{xx} + \psi_{xx})^3 \\ &= (216 + \mathrm{tr} \phi_{xx}^3)x_n^3 + \dots\end{aligned}$$

where the dots stands for the degrees of x_n lower than 3. On the other hand, the coefficient of x_n^3 in $\mathrm{tr} \mathrm{Hess}^3 f$ can be read out from the cubic trace identity $\mathrm{tr} \mathrm{Hess}^3 f = \alpha f$ which yields $\alpha = 216 + \mathrm{tr} \phi_{xx}^3$. Using (3.1) and (3.2),

$$\alpha = -216 + (-216n_1 - 27n_2 + 27n_3) = 162(1 - n_1) \equiv 3\lambda(n_1 - 1),$$

which yields $n_1 = \frac{\alpha}{3\lambda} + 1$, as required.

Finally, note that the dimension n_1 is uniquely determined from the cubic trace identity by virtue of λ and α , and in view of the remark made above the ratio $\frac{\alpha}{3\lambda}$ is a congruence invariant. Since n_2 is determined by $n_2 = \frac{n+1-3n_1}{2}$, we conclude that it is also a congruence invariant. The proposition is proved completely. \square

5.1. The cubic trace identity in the Clifford case.

Proposition 5.1. *Let $\mathcal{A} = (A_0, \dots, A_q) \in \mathrm{Cliff}(\mathbb{R}^{2m}, q)$ and*

$$C_{\mathcal{A}}(x) = \sum_{i=0}^q \langle y, A_i y \rangle x_{i+1}, \quad y = (x_{q+2}, \dots, x_{q+1+2m}) \in \mathbb{R}^{2m}. \quad (5.5)$$

be the Clifford eigencubic associated with \mathcal{A} . Then

$$\sigma_2(C_{\mathcal{A}}) = mz^2 + (q+1)y^2, \quad z = (x_1, \dots, x_{1+q}), \quad (5.6)$$

and $C_{\mathcal{A}}$ possess the cubic trace identity with $\alpha = 3(q-1)\lambda$. In particular, $C_{\mathcal{A}}$ has the type

$$(n_1, n_2) = (q, m+1-q). \quad (5.7)$$

PROOF. Write the Hessian matrix of $f \equiv C_{\mathcal{A}}$ in the block form

$$\mathrm{Hess} f \equiv \begin{pmatrix} f_{yy} & f_{yz} \\ f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 2A_z & B \\ B^\top & 0 \end{pmatrix},$$

where B is the matrix with entries $B_{ij} = f_{y_i z_j} = 2e_i^\top A_j y$ and $\{e_i\}_{i=1}^{2m}$ is the standard basis in \mathbb{R}^{2m} . Then

$$\begin{aligned}\mathrm{tr} \mathrm{Hess}^2 f &= 4\mathrm{tr} A_z^2 + 2\mathrm{tr} BB^\top = 8mz^2 + 8 \sum_{i=1}^{2m} \sum_{j=0}^q y^\top A_j e_i e_i^\top A_j y \\ &= 8(mz^2 + (q+1)y^2).\end{aligned}$$

By Theorem 3.2 in [T2] we have $\lambda(C_{\mathcal{A}}) = -8$ which yields (5.6).

In order to establish the cubic trace identity, we note that $A_z^2 = z^2 \mathbf{1}_{\mathbb{R}^{2m}}$ and $\mathrm{tr} A_i = 0$, so that $\mathrm{tr} A_z^3 = \mathrm{tr} z^2 A_z = 0$ and

$$\mathrm{tr} \mathrm{Hess}^3 f = 8\mathrm{tr} A_z^3 + 6\mathrm{tr} A_z BB^\top = 6\mathrm{tr} A_z BB^\top. \quad (5.8)$$

From (??) $A_z A_k + A_k A_z = 2z_k \mathbf{1}_{\mathbb{R}^{2m}}$, hence

$$\begin{aligned}
 \operatorname{tr} A_z B B^\top &= \sum_{i,j=1}^{2m} (A_z)_{ij} (B B^\top)_{ij} = 4 \sum_{k=0}^q \sum_{i,j=1}^{2m} (A_z)_{ij} e_i^\top A_k y \cdot e_j^\top A_k y \\
 &= 4 \sum_{k=0}^q y^\top A_k \left(\sum_{i,j=1}^{2m} (A_z)_{ij} e_i \cdot e_j^\top \right) A_k y = 4 \sum_{k=0}^q y^\top A_k A_z A_k y \\
 &= 4 \sum_{k=0}^q y^\top (2z_k \mathbf{1}_{\mathbb{R}^{2m}} - A_z A_k) A_k y \\
 &= 4(1-q) y^\top A_z y.
 \end{aligned} \tag{5.9}$$

Combining (5.8) and (5.9) yields

$$\operatorname{tr} \operatorname{Hess}^3 f = 24(1-q)f,$$

hence f possess the cubic trace identity with $\alpha = 24(1-q) \equiv 3(q-1)\lambda$. Then by using (ii) in Lemma 5.1 we find from $n = 3n_1 + 2n_2 - 1 = q + 1 + 2m$ that $n_1 = q$ and $n_2 = m + 1 - q$. \square

Corollary 5.1. *A pair (n_1, n_2) of non-negative integers is the type of some eigencubic of Clifford type if and only if*

$$n_1 \leq \rho(n_2 + n_1 - 1). \tag{5.10}$$

PROOF. Let us first suppose that (5.10) holds. We may assume without loss of generality that $n_1 \geq 0$ and $n_2 + n_1 - 1 \geq 1$. Then setting $q = n_1$ and $m = n_1 + n_2 - 1$, the inequality $n_1 - 1 < \rho(n_2 + n_1 - 1)$ becomes equivalent to $q \leq \rho(m)$ which implies that there exists a symmetric Clifford system $\mathcal{A} = \{A_0, \dots, A_q\} \in \operatorname{Cliff}(\mathbb{R}^{2m}, q)$. Let $C_{\mathcal{A}}$ the associated with \mathcal{A} Clifford eigencubic (1.9). Then Proposition 5.1 yields that $C_{\mathcal{A}}$ is a Clifford eigencubic of type $(q, m + 1 - q) \equiv (n_1, n_2)$.

In the converse direction, let us assume there exists a Clifford eigencubic f of the type (n_1, n_2) . Since the case is trivial, we may assume that $n_1 \geq 1$. By Proposition 3.3 we have $n_1 - 1 \leq \rho(n_2 + n_1 - 1)$, hence it suffices to show that the equality in the latter inequality is impossible. We assume the contrary, i.e. that there exists a Clifford eigencubic f of the type (n_1, n_2) with $n_1 \geq 1$ and $n_1 - 1 = \rho(n_2 + n_1 - 1)$. By Proposition 5.1, f is of type $(q, m + 1 - q) = (n_1, n_2)$, hence $n_1 = q$ and $n_2 = m + 1 - q$. On the other hand, the existence of the Clifford eigencubic f implies $q \leq \rho(m)$, hence $n_1 \leq \rho(n_2 + n_1 - 1)$, a contradiction. \square

5.2. The trace identities for exceptional eigencubics.

Proposition 5.2. *Let f be an arbitrary radial eigencubic having the normal form of type (n_1, n_2) , $n_2 \in \{0, 5, 8, 14, 26\}$, and such that $\Delta\psi_{030} = 0$. Then*

$$\operatorname{tr} \operatorname{Hess}^2(f) = -\frac{1}{3}(3n_1 + n_2 + 1)\lambda x^2, \tag{5.11}$$

$$\operatorname{tr} \operatorname{Hess}^3(f) = 3(n_1 - 1)\lambda f, \tag{5.12}$$

where $L(f) = \lambda x^2 f$. In particular, the trace identities (5.11) and (5.12) are valid for any exceptional eigencubic.

Remark 5.1. Observe, that the statement of Proposition 5.2 establishes also the implications (a) \Rightarrow (c) and (b) \Rightarrow (c) in Theorem 4.

PROOF. Observe that it suffices to prove the first identity. Indeed, if (5.11) holds then by Lemma 5.1 we have $\operatorname{tr} \operatorname{Hess}^3(f) = \alpha f$ with

$$\alpha = -(n+6)\lambda + 2(3n_1 + n_2 + 1)\lambda = 3(1 - n_1)\lambda,$$

so that (5.12) follows.

We split the proof of (5.11) into two steps. First we suppose that $n_2 = 0$. Then by Proposition 3.4 f is an isoparametric eigencubic of type $(\ell + 1, 0)$, $\ell = 1, 2, 4, 8$. In particular, f is harmonic and satisfies the eiconal equation $|\nabla f|^2 = cx^4$ for some $c > 0$. Hence

$$L(f) = |\nabla f|^2 \Delta f - \frac{1}{2} \langle \nabla |\nabla f|^2, \nabla f \rangle = -6cfx^2,$$

hence $\lambda(f) = -6c$. On the other hand, by using the harmonicity of f again, we get

$$\text{tr Hess}^2(f) = \sum_{i,j=1}^n f_{x_i x_j}^2 \equiv \sum_{i=1}^n f_{x_i} \Delta f_{x_i} + \sum_{i,j=1}^n f_{x_i x_j}^2 = \frac{1}{2} \Delta |\nabla f|^2 = 2c(n+2)x^2.$$

Since $n = 3\ell + 2 \equiv 3n_1 + n_2 - 1$, we latter identity is equivalent to (5.11).

Now suppose that $n_2 = 3\ell + 2$, $\ell \in \{1, 2, 4, 8\}$, and f is written in the normal form (3.7) with $\Delta \psi_{030} = 0$. Let $\mathbb{R}^n = V_0 \oplus V_1 \oplus V_2 \oplus V_3$, $V_0 = \text{span}(e_n)$, be the associated with (3.7) orthogonal decomposition. Then

$$\text{tr Hess}^2 f = \sum_{i,j=0}^3 \text{tr } f_{V_i V_j} f_{V_j V_i} \equiv \sum_{i,j=0}^3 T_{ij}, \quad T_{ij} = T_{ji}.$$

Here $f_{V_i V_j}$ stands for the submatrix of the Hessian of f with entries $f_{u,v}$, where u and v run orthogonal coordinates in V_i and V_j respectively. We have $f_{x_n x_n} = 6x_n$, $f_{x_n \xi} = -6\xi^\top$, $f_{x_n \eta} = -3\eta^\top$, $f_{x_n \zeta} = 3\zeta^\top$, $f_{\xi \xi} = -6x_n \mathbf{1}_{V_1}$, $f_{\xi_i \eta_j} = 3e_i^\top N_j \zeta$, where $\{e_i\}$ is an orthonormal basis in V_1 , and the matrices N_i are defined by (3.17). This yields

$$\begin{aligned} T_{00} &= 36x_n^2, & T_{01} &= 36\xi^2, & T_{03} &= 9\eta^2, \\ T_{04} &= 9\zeta^2, & T_{11} &= 36n_1 x_n^2. \end{aligned}$$

We also have

$$T_{12} = 9 \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \zeta^\top N_j^\top e_i \cdot e_i^\top N_j \zeta = 9\zeta^\top \left(\sum_{j=1}^{n_2} N_j^\top N_j \right) \zeta.$$

On the other hand, since $\text{tr } N_\eta N_\eta^\top = n_1 \eta^2$ by virtue of (3.20) and $\sum_{i=1}^{n_1} Q_i N_\eta^\top e_i = \frac{1}{2} \Delta_\xi (Q_\xi N_\eta^\top \xi) = 0$ by virtue of (3.21), we find

$$\begin{aligned} T_{13} &= 9 \sum_{i=1}^{n_1} \sum_{k=1}^{n_3} (e_i^\top N_\eta e_k + \sqrt{3} \zeta^\top Q_i e_k)^2 = 9 \text{tr } N_\eta N_\eta^\top + 18\sqrt{3} \zeta^\top \sum_{i=1}^{n_1} Q_i N_\eta^\top e_i + 27\zeta^\top \left(\sum_{i=1}^{n_1} Q_i^2 \right) \zeta \\ &= 9n_1 \eta^2 + 27\zeta^\top \left(\sum_{i=1}^{n_1} Q_i^2 \right) \zeta, \end{aligned}$$

where $\{e_k\}$ is an orthonormal basis in V_3 .

By the above, ψ_{030} is harmonic, hence (3.12) yields

$$\text{tr}((\psi_{030})_{\eta\eta})^2 = \frac{1}{2} \Delta_\eta (\psi_{030})_\eta^2 = \frac{9}{4} \Delta_\eta \eta^4 = 9(n_2 + 2)\eta^2,$$

hence

$$T_{22} = \text{tr}((\psi_{030})_{\eta\eta} - 3x_n \mathbf{1}_{V_2})^2 = 9(n_2 + 2)\eta^2 + 9n_2 x_n^2.$$

Next, $f_{\eta_j \zeta_k} = 3(\xi N_j + \sqrt{2} \zeta^\top R_j) e_k$ yields

$$T_{23} = 9 \sum_{j=1}^{n_2} |N_j^\top \xi + \sqrt{2} R_j \zeta|^2 = 9 \sum_{j=1}^{n_2} (\xi^\top N_i N_i^\top \xi + 2\sqrt{2} \xi^\top N_i R_j \zeta + 2\zeta^\top R_i^2 \zeta).$$

From (3.20) $N_i N_i^\top = \mathbf{1}_{V_1}$. On the other hand, $\Delta_\eta H_i = \frac{\sqrt{2}}{3} \partial_{\eta_i} \Delta_\eta \psi_{030} = 0$, hence (3.24) yields $0 = \Delta_\eta(N_H + N_\eta R_\eta) = \sum_{j=1}^{n_2} N_j R_j$, thus,

$$T_{23} = 9n_2 \xi^2 + 18 \sum_{j=1}^{n_2} \zeta^\top R_j^2 \zeta.$$

Finally, $f_{\zeta_i \zeta_j} = 3x_0 \delta_{ij} + 3\sqrt{3} Q_\xi + 3\sqrt{2} R_\eta$ yields

$$T_{33} = 9n_3 x_0^2 + 18\sqrt{3} \operatorname{tr} Q_\xi + 18\sqrt{2} \operatorname{tr} R_\eta + 27 \operatorname{tr} Q_\xi^2 + 18\sqrt{6} \operatorname{tr} Q_\xi R_\eta + 18 \operatorname{tr} R_\eta^2.$$

From (3.27)

$$n_3 \xi^2 = \operatorname{tr}(P_\xi^\top P_\xi + Q_\xi^2) \operatorname{tr} P_\xi P_\xi^\top + \operatorname{tr} Q_\xi^2 = n_2 \xi^2 + \operatorname{tr} Q_\xi^2,$$

hence $\operatorname{tr} Q_\xi^2 = (n_3 - n_2) \xi^2 = 2(n_1 - 1) \xi^2$. Similarly, the harmonicity of ψ_{030} yields from (3.4) $\operatorname{tr} R_i = \operatorname{tr} R_j = 0$, hence taking the trace in (3.25)

$$0 = 2 \operatorname{tr} N_\eta^\top N_\eta - 2 \operatorname{tr} R_\eta^2 + \eta^2 \operatorname{tr} \mathbf{1}_{V_3} = (2n_1 + n_3) \eta^2 - 2 \operatorname{tr} R_\eta^2,$$

which yields

$$T_{33} = 9n_3 x_0^2 + 54(n_1 - 1) \xi^2 + 18\sqrt{6} \operatorname{tr} Q_\xi R_\eta + 9(4n_1 + n_2 - 2) \eta^2.$$

Summing up the found relations, we obtain

$$\begin{aligned} \sum_{i,j=0}^3 T_{ij} = & 18(3n_1 + n_2 + 1)(x_n^2 + \xi^2 + \eta^2) + 18\zeta^2 + 18\zeta^\top \left(\sum_{j=1}^{n_2} N_j^\top N_j \right) \zeta \\ & + 54\zeta^\top \left(\sum_{i=1}^{n_1} Q_i^2 \right) \zeta + 36 \sum_{j=1}^{n_2} \zeta^\top R_j^2 \zeta + 18\sqrt{6} \operatorname{tr} Q_\xi R_\eta. \end{aligned}$$

Applying the ξ -Laplacian to (3.19) and the ζ -Laplacian to (3.22), and taking into account that $\operatorname{tr} Q_i = \operatorname{tr} R_j = 0$, we find respectively

$$\zeta^\top \left(\sum_{j=1}^{n_2} N_j^\top N_j + \sum_{i=1}^{n_1} Q_i^2 \right) \zeta = n_1 \zeta^2, \quad \zeta^\top \left(\sum_{j=1}^{n_2} R_j^2 + \sum_{i=1}^{n_1} Q_i^2 \right) \zeta = \frac{n_3 + 2}{2} \zeta^2,$$

which yields

$$18\zeta^\top \left(\sum_{j=1}^{n_2} N_j^\top N_j \right) \zeta + 54\zeta^\top \left(\sum_{i=1}^{n_1} Q_i^2 \right) \zeta + 36 \sum_{j=1}^{n_2} \zeta^\top R_j^2 \zeta = 18(n_1 + n_3 + 2) \zeta^2.$$

Thus,

$$\sum_{i,j=0}^3 T_{ij} = 18(3n_1 + n_2 + 1)(x_n^2 + \xi^2 + \eta^2 + \zeta^2) + 18\sqrt{6} \operatorname{tr} Q_\xi R_\eta.$$

In order to prove (5.11) it remains to show only that $\operatorname{tr} Q_\xi R_\eta = 0$, or equivalently that $\operatorname{tr} Q_i R_j = 0$ for all i, j . To this end, let us fix an index i , $1 \leq i \leq n_1$. Then (3.27) yields $Q_\xi^3 = \xi^2 Q_\xi$, hence $Q_i^3 = Q_i$. This means that Q_i has three eigenvalues: ± 1 and 0. Let W_\pm and W_0 denote the corresponding eigenspaces and let $\zeta = (w_+, w_-, w_0)$ be the vector decomposition corresponding to the decomposition $\zeta \in V_3 = W_+ \oplus W_- \oplus W_0$. Then (3.22) yields

$$\sum_{k=1, k \neq i}^{n_1} (\zeta^\top Q_k \zeta)^2 + \sum_{j=1}^{n_2} (\zeta^\top R_j \zeta)^2 = \zeta^4 - (w_+^2 - w_-^2)^2 = 4w_+^2 w_-^2 + 2w_0^2 (w_+^2 + w_-^2) + w_0^4,$$

implying that R_j has the following block structure:

$$R_j = \begin{pmatrix} \mathbf{0}_{W_+} & * & * \\ * & \mathbf{0}_{W_-} & * \\ * & * & M_j \end{pmatrix},$$

where M_j is a symmetric endomorphism of W_0 with $\text{tr } M_j = \text{tr } R_j = 0$. We have

$$Q_i R_j = \begin{pmatrix} \mathbf{0}_{W_+} & * & * \\ * & \mathbf{0}_{W_-} & * \\ * & * & -\frac{1}{2}M_j \end{pmatrix},$$

which yields that $\text{tr } Q_i R_j = -\frac{1}{2} \text{tr } M_j = 0$. This finishes the proof of the theorem. \square

5.3. Completion of the proof of Theorem 3 and Theorem 4. Now we are able to finish the proof of Theorem 4. The following two corollaries establish the implications (c) \Rightarrow (a) and (b) \Rightarrow (a) in Theorem 4, respectively. Furthermore, Corollary 5.3 also establishes the ‘only-if’ part in Theorem 3.

Corollary 5.2. *If a radial eigencubic f possess the quadratic trace identity and $n_2 \in \{0, 5, 8, 14, 26\}$ then f is exceptional.*

PROOF. We argue by contradiction and assume that there is an eigencubic f of Clifford type satisfying the quadratic trace identity. Then $\sigma_2(f)$ has a single eigenvalue, and it follows from Proposition 5.1 that for the associated to f Clifford system \mathcal{A} the equality $m = q + 1$ holds. But in that case (5.7) yields $n_2 = m + 1 - q = 2$, a contradiction. \square

Corollary 5.3. *Let f be a radial eigencubic in the normal form (3.7). If ψ_{030} is either identically zero or irreducible then f is an exceptional eigencubic. Equivalently, if f is a radial eigencubic of Clifford type then $\psi_{030} \not\equiv 0$ and reducible.*

PROOF. If $\psi_{030} \equiv 0$ then in view of (3.12) we have $n_2 = 0$, hence by Proposition 3.4 f is exceptional. If $\psi_{030} \not\equiv 0$ and irreducible then by the Eiconal Cubic Theorem, $n_2 = 3\ell + 2$, $\ell = 1, 2, 4, 8$, and $\Delta\psi_{030} = 0$, in particular Proposition 5.2 yields that f possess the quadratic trace identity. If $n \neq 3m$, $m \in \{1, 2, 4, 8\}$ then by Corollary 5.2, f is exceptional. Now suppose that $n = 3k$, where $k \in \{1, 2, 4, 8\}$, and assume that f is an eigencubic of Clifford type. Then we have from (3.2) that $n_1 = k - 2\ell - 1$ and (5.11) yields

$$\text{tr Hess}^2(f) = -\frac{1}{3}(3n_1 + n_2 + 1)\lambda x^2 = (\ell - k)\lambda x^2,$$

hence

$$\sigma_2(f) = -\frac{1}{\lambda} \text{tr Hess}^2(f) = (k - \ell)x^2.$$

By our assumption, f an eigencubic is of Clifford type, say, given by (1.9). Then, $n = 2m + q + 1$, where by Proposition 5.1, $q = n_1 = k - 2\ell - 1$, hence $n \equiv 3k = 2(m - \ell) + k$. We find $k = m - \ell$. On the other hand, arguing as in Corollary 5.2, we see that $m = q + 1 = k - 2\ell$, hence $k = m + 2\ell$. Since $\ell \neq 0$ we get a contradiction which proves that f is an exceptional radial eigencubic. \square

6. Examples of exceptional eigencubics

In section 1.3 we already discussed the Cartan isoparametric polynomials which are also exceptional radial eigencubics of types $(\ell + 1, 0)$, $\ell = 1, 2, 4, 8$. Below we exhibit more examples of exceptional eigencubics. These examples cover all realizable types of exceptional eigencubics presented in Table 1 above.

6.1. Hsiang's trick. For our further purposes, we describe the Hsiang construction with minor modifications. Let $\mathfrak{G}'(k, \mathbb{R})$ be the vector space of quadratic forms of $k \geq 2$ real variables with trace zero. We identify $\mathfrak{G}'(k, \mathbb{R})$ with the vector space of $k \times k$ real symmetric matrices of trace zero equipped with the scalar product $\langle X, Y \rangle = \frac{1}{2} \operatorname{tr} XY$, $X, Y \in \mathfrak{G}'(k, \mathbb{R})$. Then the orthogonal group $SO(k)$ acts naturally on $\mathfrak{G}'(k, \mathbb{R})$ as substitutions, and $\mathfrak{G}'(k, \mathbb{R})$ is invariant with respect to the action. Consider an isometry $i : \mathbb{R}^N \rightarrow \mathfrak{G}'(k, \mathbb{R})$, where $N = \frac{k^2+k-2}{2}$, and let $\Gamma = i^*(SO(k))$ be the corresponding pullback of $SO(k)$ into $O(N)$. It is well-known that the coefficients of the characteristic polynomial

$$\det(X - \lambda \mathbf{1}) = \lambda^k + b_2(X)\lambda - b_3(X) + \dots + (-1)^k b_k(X), \quad X \in \mathfrak{G}'(k, \mathbb{R}),$$

form a complete set of basic invariants with respect to the $SO(k)$ action [GW]. Then the polynomial forms $\beta_k(x) = i^*b_k \in \mathbb{R}[x_1, \dots, x_N]$ are invariant under the action of the group Γ , and form a complete set of invariants:

$$\mathbb{R}[x_1, \dots, x_N]^\Gamma = \mathbb{R}[\beta_2, \dots, \beta_k] \quad (6.1)$$

(as usually, $\mathbb{R}[x_1, \dots, x_N]^\Gamma$ denotes the subring of Γ -invariant polynomials). Hsiang proves that L is an invariant operator in the sense that for any element $g \in O(N)$, the operator L commutes with the linear substitution g : $g^*L = Lg^*$. Since $\Gamma \subset O(N)$ and L is invariant, we have $L(\beta_k) \in \mathbb{R}[x_1, \dots, x_N]^\Gamma$, hence by (6.1)

$$L(\beta_k) \in \mathbb{R}[\beta_2, \dots, \beta_k]. \quad (6.2)$$

We have $b_2(X) = \frac{1}{2}((\operatorname{tr} X)^2 - \operatorname{tr} X^2) = -\frac{1}{2} \operatorname{tr} X^2$, therefore

$$\beta_2 = i^*(b_2) = -x^2.$$

On the other hand, $\deg L(\beta_3) = 5$, thus by (6.2) there are $c_1, c_2 \in \mathbb{R}$ such that

$$L(\beta_3) = c_1\beta_2\beta_3 + c_2\beta_5. \quad (6.3)$$

Now suppose $3 \leq k \leq 4$. Then $\beta_5 \equiv 0$, hence (6.3) reads as follows:

$$L(\beta_3) = c_1x^2\beta_3, \quad (6.4)$$

which is equivalent to that β_3 is a *radial* eigenfunction in \mathbb{R}^N . This yields two (irreducible) eigencubics: in \mathbb{R}^5 and in \mathbb{R}^9 , for $k = 3$ and $k = 4$ respectively. The first example is easily identified with the Cartan isoparametric cubic θ_1 in \mathbb{R}^5 , see (1.11) above. Indeed, applying the same argument to the Laplacian and the length of the gradient which obviously are invariant operators, we find in view of $\deg \Delta\beta_3 = 1$ and $\deg |\nabla\beta_3|^2 = 4$ that

$$\Delta\beta_3 = 0, \quad |\nabla\beta_3|^2 = c_3\beta_2^2 + c_4\beta_4, \quad (6.5)$$

and in view of $k = 3$ we have $\beta_4 \equiv 0$, so that (6.5) becomes equivalent to the Münzner-Cartan differential equations (1.10) for θ_1 . This can also be done explicitly if one consider a map $i : \mathbb{R}^5 \rightarrow \mathfrak{G}'(3, \mathbb{R})$ given by

$$X = i(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} x_4 - \frac{1}{\sqrt{3}}x_5 & x_3 & x_2 \\ x_3 & -x_4 - \frac{1}{\sqrt{3}}x_5 & x_1 \\ x_2 & x_1 & \frac{2}{\sqrt{3}}x_5 \end{pmatrix}, \quad x \in \mathbb{R}^5. \quad (6.6)$$

Then $\langle X, X \rangle = \frac{1}{2} \operatorname{tr} X^2 = x^2$, hence i is indeed an isometry. Hence $\beta_3 \equiv \det X$ provides an explicit determinantal representation for θ_1 (up to a constant factor). An important feature of the obtained determinantal representation (6.6) is that (in view of $\operatorname{tr} X = 0$)

$$\beta_3(x) = -\frac{1}{3} \operatorname{tr} X^3. \quad (6.7)$$

In case $k = 4$, the quartic form β_4 is no more trivial, hence (6.4) yields a non-homogeneous radial eigencubic β_3 in \mathbb{R}^9 discovered by Hsiang in [H]. In this case,

$$\beta_3 = -\frac{1}{6}(\operatorname{tr} X)^3 + \frac{1}{2} \operatorname{tr} X \operatorname{tr} X^2 - \frac{1}{3} \operatorname{tr} X^3 = -\frac{1}{3} \operatorname{tr} X^3,$$

hence β_3 possess also a trace identity like (6.7). Up to a congruence, β_3 coincides with the exceptional radial eigencubic of type $(0, 5)$ constructed in Example 6.1 below.

The above constructions can be repeated literally for the corresponding Hermitian analogues $\mathfrak{G}'(3, \mathbb{C}) \cong \mathbb{R}^8$ and $\mathfrak{G}'(4, \mathbb{C}) \cong \mathbb{R}^{15}$ with the special unitary group $SU(3)$ acting on them. This yields respectively the Cartan isoparametric cubic θ_2 of type $(3, 0)$ in \mathbb{R}^8 and the Hsiang (non-homogeneous) exceptional eigencubic of type $(0, 8)$ in \mathbb{R}^{15} . For $k = 3$ it is still possible to obtain the quaternionic (non-commutative) and octonionic (non-associative) counterparts of the above constructions by using the maximal orbits of $Sp(3)$ and F_4 instead of $SO(3)$ on the corresponding Hermitian matrices of trace zero. This yields the Cartan isoparametric eigencubics θ_4 and θ_8 , respectively. For $k = 4$, however, there is only the quaternionic counterpart in \mathbb{R}^{27} of type $(0, 14)$ (some care is needed to appropriately interpret the trace of the corresponding matrix).

6.2. The case $n_1 = 0$. The following proposition shows that all types in Table 2 with $n_1 = 0$ except for $(0, 26)$ are realizable.

Proposition 6.1. *For $n_1 = 0$, there exist only exceptional eigencubic of type $(0, 3\ell + 2)$, where $\ell = 1, 2, 4$.*

PROOF. Let us suppose that $f \in E(0, n_2)$ is an exceptional eigencubic given in the normal form (3.7). Then $n_2 \neq 0$, hence the condition $n_2 = 3\ell + 2$, $\ell = 1, 2, 4, 8$, yields $n_3 = n_2 - 2 = 3\ell$, and the condition $\dim V_1 = n_1 = 0$ yields $\psi = \psi_{012} + \psi_{030}$. Let us consider

$$\chi(\eta, \zeta) = \psi_{030} - \psi_{012}.$$

Then from (3.4) and $\Delta_\eta \psi_{030} = 0$ we see that χ is harmonic. Furthermore, by virtue of (3.11), (3.12) and (3.14)

$$|\nabla \chi|^2 = (\psi_{030})_\eta^2 - 2(\psi_{030})_\eta^\top (\psi_{012})_\eta + (\psi_{012})_\eta^2 + (\psi_{012})_\zeta^2 = \frac{9}{2}(\eta^2 + \zeta^2)^2, \quad (6.8)$$

hence $\sqrt{2}\chi$ satisfies that Cartan-Münzner equations (1.10). Since $\ell \geq 1$, the Cartan theorem implies that the dimension $\dim V_2 + \dim V_3 = 2n_2 - 2 = 6\ell + 2$ can take only values 5, 8, 14, 26. This implies $\ell \in \{1, 2, 4\}$.

In the converse direction, let us assume that $\ell \in \{1, 2, 4\}$ and show how to construct an exceptional eigencubic of type $(0, 3\ell + 2)$. To this end, we consider the division algebras $\mathbb{F}_{2\ell}$ and \mathbb{F}_ℓ . Regarding $\mathbb{F}_{2\ell}$ as the Cayley-Dickson doubling of \mathbb{F}_ℓ [Ba], we write

$$\mathbb{F}_{2\ell} = \mathbb{F}_\ell \oplus \mathbb{F}_\ell, \quad (6.9)$$

where the multiplication and conjugation on $\mathbb{F}_{2\ell}$ is given by

$$(a \oplus b)(c \oplus d) = (ac - d\bar{b}) \oplus (\bar{a}d + cb), \quad \overline{(a \oplus b)} = (\bar{a}, -b). \quad (6.10)$$

Let $\gamma_i : \mathbb{F}_{2\ell} \rightarrow \mathbb{F}_\ell$ denote the canonical projection on the i th component. By identifying \mathbb{F}_ℓ and \mathbb{R}^ℓ , we consider the induced projections

$$\tilde{\gamma}_i : \mathbb{R}^{6\ell+2} \cong \mathbb{R}^2 \oplus \mathbb{F}_{2\ell} \oplus \mathbb{F}_{2\ell} \oplus \mathbb{F}_{2\ell} \rightarrow \mathbb{R}^{3\ell+2} \cong \mathbb{R}^2 \oplus \mathbb{F}_\ell \oplus \mathbb{F}_\ell \oplus \mathbb{F}_\ell. \quad (6.11)$$

Then it follows readily from the definition of the Cartan polynomials (1.11) that $\theta_\ell = \theta_{2\ell} \circ \tilde{\gamma}_1$. Let $x = (x_1, x_2, z_1, z_2, z_3) \in \mathbb{R}^2 \oplus \mathbb{F}_{2\ell} \oplus \mathbb{F}_{2\ell} \oplus \mathbb{F}_{2\ell} = \mathbb{R}^{6\ell+2}$ and let also $z_i = z'_i \oplus z''_i$ according to (6.9). Then we denote

$$\eta := (x_1, x_2, z'_1, z'_2, z'_3) \in \mathbb{R}^{3\ell+2}, \quad \zeta := (z''_1, z''_2, z''_3) \in \mathbb{R}^{3\ell}. \quad (6.12)$$

In this notation,

$$\theta_{2\ell}(\eta, \zeta) = x_1^3 + \frac{3x_1}{2}(2|z_3|^2 - |z_1|^2 - |z_2|^2 + 2x_2^2) + \frac{3\sqrt{3}}{2}[x_2(|z_1|^2 - |z_2|^2) + \operatorname{Re} z_1 z_2 z_3],$$

and

$$\theta_\ell(\eta) = \theta_{2\ell}(\eta, 0), \quad (6.13)$$

where the real part is defined by (1.12). We claim that

$$F(\eta, \zeta, x_{3\ell+3}) := x_{3\ell+3}^3 + \frac{3}{2}(\zeta^2 - \eta^2)x_{3\ell+3} + \frac{1}{\sqrt{2}}(\theta_{2\ell}(\eta, \zeta) - 2\theta_{2\ell}(\eta, 0)). \quad (6.14)$$

is the required exceptional eigencubic of type $(0, 3\ell+2)$ in $\mathbb{R}^{3\ell+3}$. First we observe that the above expression is written in the norm form, where $\phi = \frac{3}{2}(\zeta^2 - \eta^2)$ and $\psi = \frac{\theta_{2\ell}(\eta, \zeta) - 2\theta_{2\ell}(\eta, 0)}{\sqrt{2}}$. Setting $\eta \in V_2 = \mathbb{R}^{3\ell+2}$ and $\zeta \in V_3 = \mathbb{R}^{3\ell}$, we determine the decomposition of ψ into the corresponding homogeneous parts $\eta^i \otimes \zeta^{3-i}$. To this end, we denote $\psi = \psi_{030} + \psi_{012}$, where

$$\psi_{030} = -\frac{1}{\sqrt{2}}\theta_{2\ell}(\eta, 0), \quad \psi_{012} := \frac{1}{\sqrt{2}}(\theta_{2\ell}(\eta, \zeta) - \theta_{2\ell}(\eta, 0)), \quad (6.15)$$

and notice that $\psi_{030} \in \eta \otimes \eta \otimes \eta$ by the definition and also $\psi_{012} \in \eta \otimes \zeta \otimes \zeta$. Indeed, to verify the latter identity, we write

$$\psi_{012} = \frac{3x_1}{2}(2|z_3''|^2 - |z_1''|^2 - |z_2''|^2) + \frac{3\sqrt{3}}{2}[x_2(|z_1''|^2 - |z_2''|^2) + D]$$

where $D = \operatorname{Re} z_1 z_2 z_3 - \operatorname{Re} z_1' z_2' z_3'$. Observe that the multiplication of $z_i' \in \mathbb{F}_\ell$ is associative for $\ell \leq 4$, and that in view of (6.10)

$$\operatorname{Re}(0 \oplus b) = 0, \quad b \in \mathbb{F}_\ell. \quad (6.16)$$

Then (6.10) yields

$$\operatorname{Re}(z_1' \oplus z_1'') \cdot (z_2' \oplus z_2'') \cdot (z_3' \oplus z_3'') \operatorname{Re}(z_1' z_2' z_3' - z_2'' z_1'' z_3' - z_3'' z_1'' z_2' - z_3'' z_2'' z_1'),$$

thus

$$D = -\operatorname{Re}(z_2'' z_1'' z_3' + z_3'' z_1'' z_2' + z_3'' z_2'' z_1') \in \eta \otimes \zeta^2,$$

which, in vie of (6.12), yields $\psi_{012} \in \eta \otimes \zeta^2$.

In order to show that F is an eigencubic, we need by Proposition 3.1 to verify that ϕ and ψ satisfy equations (3.5) and (3.6). The former equation is equivalent to the system in Proposition 3.2, where equations (3.8)–(3.10), (3.13) are trivial because $n_1 = 0$, equation (3.12) is satisfied by our choice of ψ_{030} in (6.15):

$$(\psi_{030})_\eta^2 = \frac{9}{2}|\nabla\theta_\ell(\eta)| = \frac{9}{2}\eta^4, \quad (6.17)$$

and equations (3.11) and (3.14) have the following form:

$$6(\psi_{012})_\eta^2 = 27\zeta^4, \quad (6.18)$$

and

$$2(\psi_{030})_\eta^\top (\psi_{012})_\eta - (\psi_{012})_\zeta^2 = -9\zeta^2\eta^2, \quad (6.19)$$

respectively. But these equations follows immediately by collecting the terms by homogeneity in the identity

$$\frac{9}{2}(\eta^2 + \zeta^2)^2 = (\psi_{012})_\eta^2 - 2(\psi_{012})_\eta^\top (\psi_{030})_\eta + (\psi_{030})_\eta^2 + (\psi_{012})_\zeta^2,$$

which is obtained from $|\nabla\theta_{2\ell}(\eta, \zeta)|^2 = 9(\eta^2 + \zeta^2)^2$, where $\theta_{2\ell}(\eta, \zeta) = \sqrt{2}(\psi_{012} - \psi_{030})$ by virtue of (6.15).

Thus, it remains only to verify (3.6). To this end, we note that by (6.19), (6.18) and (6.17)

$$\begin{aligned} |\nabla\psi|^2 &= (\psi_{012})_\zeta^2 + (\psi_{012})_\eta^2 + 2(\psi_{012})_\eta^\top (\psi_{030})_\eta + (\psi_{030})_\eta^2 \\ &= \frac{9}{2}(\eta^2 + \zeta^2)^2 + 4(\psi_{012})_\eta^\top (\psi_{030})_\eta, \end{aligned}$$

hence

$$\begin{aligned} \nabla\psi \cdot \nabla(|\nabla\psi|^2) &= 54\psi(\eta^2 + \zeta^2) + 4\psi_\zeta^\top (\psi_{012})_\zeta \eta (\psi_{030})_\eta + 4\psi_\eta^\top (\psi_{030})_\eta \eta (\psi_{012})_\eta \\ &= 54\psi(\eta^2 + \zeta^2) + 4(\psi_{012})_\zeta^\top (\psi_{012})_\zeta \eta (\psi_{030})_\eta + 4(\psi_{012} + \psi_{030})_\eta^\top (\psi_{030})_\eta \eta (\psi_{012})_\eta \end{aligned}$$

We have by (6.19) and (6.17)

$$\begin{aligned}
4(\psi_{012})_\zeta^\top (\psi_{012})_{\zeta\eta} (\psi_{030})_\eta 2((\psi_{012})_\zeta^2)^\top (\psi_{030})_\eta \\
&= 2(2(\psi_{030})_\eta^\top (\psi_{012})_\eta + 9\zeta^2\eta^2)^\top (\psi_{030})_\eta \\
&= 4(\psi_{012})_\eta^\top (\psi_{030})_{\eta\eta} (\psi_{030})_\eta + 108\zeta^2\psi_{030} \\
&= 2(\psi_{012})_\eta^\top ((\psi_{030})_\eta^2)_\eta + 108\zeta^2\psi_{030} \\
&= 36\eta^2\psi_{012} + 108\zeta^2\psi_{030}
\end{aligned}$$

and similarly

$$4(\psi_{030})_\eta^\top (\psi_{030})_{\eta\eta} (\psi_{012})_\eta = 36\eta^2\psi_{012}.$$

We also have by (6.19) and (6.18)

$$\begin{aligned}
4(\psi_{012})_\eta^\top (\psi_{030})_{\eta\eta} (\psi_{012})_\eta &= 4((\psi_{012})_\eta^\top (\psi_{030})_\eta)^\top (\psi_{012})_\eta \\
&= 2(-9\zeta^2\eta^2 + (\psi_{012})_\zeta^2)^\top (\psi_{012})_\eta \\
&= -36\zeta^2\psi_{012} + 4(\psi_{012})_\zeta^\top (\psi_{012})_{\zeta\eta} (\psi_{012})_\eta \\
&= -36\zeta^2\psi_{012} + 2(\psi_{012})_\zeta^\top ((\psi_{012})_\eta^2)_\zeta \\
&= 36\zeta^2\psi_{012}
\end{aligned}$$

Combining the found identities we obtain

$$\begin{aligned}
\psi_{\bar{x}}^\top \psi_{\bar{x}\bar{x}} \psi_{\bar{x}} &\equiv \frac{1}{2} \nabla \psi \cdot \nabla (|\nabla \psi|^2) = 27\psi + 36\eta^2\psi_{012} + 54\zeta^2\psi_{030} + 18\zeta^2\psi_{012} \\
&= 27\psi_{030}(\eta^2 + 3\zeta^2) + 9\psi_{030}(7\eta^2 + 5\zeta^2)
\end{aligned}$$

Furthermore,

$$2\phi \phi_{\bar{x}}^\top \psi_{\bar{x}} = 9(\zeta^2 - \eta^2)(\zeta^\top \psi_\zeta - \eta^\top \psi_\eta)9(\zeta^2 - \eta^2)(\psi_{012} - 3\psi_{030}),$$

which finally yields

$$2\phi \phi_{\bar{x}}^\top \psi_{\bar{x}} + \psi_{\bar{x}}^\top \psi_{\bar{x}\bar{x}} \psi_{\bar{x}} = 54(\eta^2 + \zeta^2)\psi$$

and proves (3.6). Thus, F is indeed a radial eigencubic. From (6.14) we see that $(n_1, n_2) = (0, 3\ell + 2)$. On the other hand, by our construction, $\Delta\psi_{030} = \frac{1}{\sqrt{2}}\Delta\theta_\ell(\eta) = 0$, hence Corollary 5.3 yields that F is exceptional. The proposition is proved completely. \square

Example 6.1. Let us consider the following cubic polynomial in \mathbb{R}^9 given by

$$f(x) := \operatorname{Re} \prod_{s=1}^3 (x_{3s-2}\mathbf{i} + x_{3s-1}\mathbf{j} + x_{3s}\mathbf{k}) \equiv \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is the standard basis elements in quaternion algebra $\mathbb{H} = \mathbb{F}_4$. Then it is straightforward to see that f satisfies (1.4) with $\lambda = -2$. Observe also that f satisfies the quadratic trace identity

$$\sigma_2(f) = 2 \sum_{i=1}^9 x_i^2, \quad (6.20)$$

hence, by Corollary 5.2, f is an exceptional eigencubic. From (5.11) and (6.20), $3n_1 + n_2 + 1 = 6$ and in virtue of (3.2) $9 = n = 3n_1 + 2n_2 - 1$, hence $n_1 = 0$ and $n_2 = 5$. Thus f is an exceptional eigencubic of type $(0, 5)$. It was already mentioned in section 6.1 that f is congruent to the Hsiang example in \mathbb{R}^9 mentioned in [H].

6.3. The case $n_1 = 1$.

Proposition 6.2. *If $n_1 = 1$ then there are exactly four (congruence classes of) exceptional eigencubics of types $E(1, 3\ell + 2)$, $\ell = 1, 2, 4, 8$. Any such a cubic is congruent to*

$$f(t) = [t_{3\ell+3}^3 - \frac{3}{2}t_{3\ell+3}(t_1^2 + \dots + t_{3\ell+2}^2) + \frac{1}{\sqrt{2}}\theta_\ell(t_1, \dots, t_{3\ell+2})]_{\mathbb{C}}, \quad t \in \mathbb{R}^{3\ell+3},$$

where

$$[g]_{\mathbb{C}}(x, y) := \frac{1}{2}(g(x + iy) + g(x - iy)), \quad x, y \in \mathbb{R}^{3\ell+3}$$

is the complex doubling of a polynomial g .

PROOF. We assume that f is given in the normal form (3.7). By the assumption $\dim V_1 = 1$, hence $n_3 = n_2$. Since f is exceptional, we also have $n_2 = 3\ell + 2$, where $\ell = 1, 2, 4, 8$, see Table 2. Furthermore, by identifying the coefficient of ξ_1^2 in the first identity in (3.27) one finds

$$P_1^\top P_1 + Q_1^2 = \mathbf{1}_{V_3}, \quad P_1 P_1^\top = \mathbf{1}_{V_2},$$

which yields $\text{tr } Q_1^2 = n_3 - n_2 = 0$, hence $Q_1 = 0$. Thus $\psi_{102} \equiv 0$.

In the notation of section 3, $\psi_{111} = 3\xi_1 N_\eta \zeta$, where N_i , $1 \leq i \leq 3\ell + 2$, are matrices of size $1 \times (3\ell + 2)$. By (3.19) and (3.20) we see that $\{N_1^\top, \dots, N_{3\ell+2}^\top\}$ forms an orthonormal basis in $V_3 = \mathbb{R}^{3\ell+2}$. Using the freedom to choose the orthonormal basis elements, we can ensure that $N_i^\top = e_i$, where $\{e_i\}$ is the standard orthonormal basis in $\mathbb{R}^{3\ell+2}$. This yields

$$N_\eta = N_1^\top \eta_1 + \dots + N_{n_2}^\top \eta_{n_2} = \eta^\top, \quad \eta \in V_2,$$

hence $\psi_{111} = 3\xi_1 \eta^\top \zeta$ and we have for the normal form

$$f = x_n^3 - 3\xi_1 x_n^2 + \frac{3}{2}(\zeta^2 - \eta^2)x_n + 3\xi_1 \eta^\top \zeta + \psi_{030} + \psi_{012} \quad (6.21)$$

Moreover, in this notation (3.24) reads as

$$H^\top = -\eta^\top R_\eta, \quad (6.22)$$

where $H(\eta) = \frac{\sqrt{2}}{3}\nabla_\eta \psi_{030}(\eta)$ and $\psi_{012} = \frac{3\sqrt{2}}{2}\zeta^\top R_\eta \zeta$. This yields

$$\frac{\sqrt{2}}{3}(\psi_{030})_{\eta_i \eta_j} = (H_i)_{\eta_j} = (H_i)_{\eta_j},$$

and

$$\eta^\top R_\eta e_i + e_i^\top R_\eta e_i = \eta^\top R_\eta e_j + e_i^\top R_\eta e_j.$$

By virtue of symmetricity of matrix R_η ,

$$\frac{1}{2}(\partial_{\eta_j} r_i - \partial_{\eta_i} r_j) = \eta^\top (R_i e_j - R_j e_i) = 0, \quad (6.23)$$

where $r_i = \eta^\top R_i \eta$. The latter relation yields that there exists a homogeneous cubic polynomial $r = r(\eta)$ such that $r_i = \partial_{\eta_i} r$. By the homogeneity of r and (6.22) we have

$$r(\eta) = \frac{1}{3} \sum_{i=1}^{n_2} \eta_i \partial_{\eta_i} r \equiv \frac{1}{3} \sum_{i=1}^{n_2} \eta_i r_i = \frac{1}{3} \eta^\top R_\eta \eta = -\frac{1}{3} H^\top \eta = -\frac{\sqrt{2}}{9} (\psi_{030})_\eta^\top \eta = -\frac{\sqrt{2}}{3} \psi_{030}.$$

This yields, in particular,

$$\psi_{030} = -\frac{1}{\sqrt{2}} \eta^\top R_\eta \eta. \quad (6.24)$$

By choosing an orthonormal basis in $V_2 = \mathbb{R}^{3\ell+2}$ we may ensure (in view of the Eiconal Cubic Theorem and (3.11)) that $\psi_{030}(\eta) = \frac{1}{\sqrt{2}}\theta_\ell(\eta)$, where $\theta_\ell(\eta)$ is the Cartan polynomial (1.11). Then (6.24) becomes $\eta^\top R_\eta \eta = -\theta_\ell(\eta)$. Setting η by $\eta \pm i\zeta$ in the latter identity, where $i^2 = -1$, and summing we get

$$\frac{1}{2}(\theta_\ell(\eta + i\zeta) + \theta_\ell(\eta - i\zeta)) = -\eta^\top R_\eta \eta + 2\zeta^\top R_\zeta \eta + \zeta^\top R_\eta \zeta.$$

From (6.23), $\zeta^\top R_i e_j = \zeta^\top R_j e_i$, hence $\zeta^\top R_\eta \zeta = \zeta^\top R_\zeta \eta$, and the above relation yields

$$\frac{1}{2}(\theta_\ell(\eta + i\zeta) + \theta_\ell(\eta - i\zeta)) = \sqrt{2}\psi_{030} + 3\zeta^\top R_\eta \zeta \equiv \sqrt{2}(\psi_{030} + \psi_{012}). \quad (6.25)$$

Now let us define a new cubic polynomial

$$g(t) := t_{3\ell+3}^3 - \frac{3}{2}t_{3\ell+3}(t_1^2 + \dots + t_{3\ell+2}^2) + \frac{1}{\sqrt{2}}\theta_\ell(t_1, \dots, t_{3\ell+2}).$$

Then, setting $t_\pm = (\eta_1 \pm \zeta_1 i, \dots, \eta_{3\ell+2} \pm \zeta_{3\ell+2} i, x_n \pm \xi i)$, one easily finds that by virtue of (6.25) and (6.21) that

$$\begin{aligned} [g]_{\mathbb{C}} &\equiv \frac{1}{2}(g(t_-) + g(t_+)) \\ &= x_n^3 - 3\xi_1(x_n^2 - \eta^\top \zeta) + \frac{3}{2}(\zeta^2 - \eta^2)x_n + \frac{1}{2\sqrt{2}}(\theta_\ell(\eta + i\zeta) + \theta_\ell(\eta - i\zeta)) \\ &= f, \end{aligned}$$

which yields the required representation and proves the proposition. \square

6.4. An exceptional eigencubic of type (4, 5) in \mathbb{R}^{21} . Now we exhibit an example of an exceptional eigencubic of type $E(4, 5)$ in \mathbb{R}^{21} . Let $o_0 = 1, o_1, \dots, o_7$ be a basis for the octonion algebra, $\mathbb{F}_8 = \mathbb{O}$. The multiplication table is given by $o_i \cdot o_{i+1} = o_{i+3}$, where the indices are permuted cyclically and translated modulo 7, see for instance [Ba]. For any vector $u = (u_1, \dots, u_7) \in \mathbb{R}^7$ we denote by o_u the imaginary octonion $o_u = u_1 o_1 + \dots + u_7 o_7$. Let us consider the cubic form

$$f \equiv f_{\mathbb{F}_8} = \operatorname{Re} o_u(o_v o_w), \quad x = (u, v, w) \in \mathbb{R}^{21}, \quad (6.26)$$

where $u = (x_1, \dots, x_7)$, $v = (x_8, \dots, x_{14})$, $w = (x_{15}, \dots, x_{21})$. We shall need the following known properties of the real part (see, for instance, Corollary 15.12 in [Ad]): for any three octonions $\alpha, \beta, \gamma \in \mathbb{F}_8$

$$\operatorname{Re} \alpha \beta = \operatorname{Re} \beta \alpha, \quad (6.27)$$

$$\operatorname{Re}(\alpha \beta) \gamma = \operatorname{Re} \alpha (\beta \gamma), \quad (6.28)$$

$$\operatorname{Re}(\alpha \beta) \gamma = \operatorname{Re}(\beta \gamma) \alpha. \quad (6.29)$$

Since $\operatorname{Re} \alpha = \operatorname{Re} \bar{\alpha}$, where $\bar{\alpha}$ denotes the conjugate octonion, we have for *imaginary* octonions

$$\operatorname{Re} o_u(o_v o_w) = \operatorname{Re} \overline{o_u(o_v o_w)} = \operatorname{Re} (\overline{o_v o_w}) \bar{o}_u = -\operatorname{Re}(o_w o_v) o_u = -\operatorname{Re}(o_w o_v) o_u.$$

This yields by virtue of (6.29) and (6.28)

$$\operatorname{Re} o_u(o_v o_w) = -\operatorname{Re} o_u(o_w o_v), \quad (6.30)$$

i.e. the cubic form f is an alternating function in the three variables.

Proposition 6.3. *The cubic form $f = \operatorname{Re} o_u(o_v o_w)$ is an exceptional eigencubic in \mathbb{R}^{21} of class $E(4, 5)$.*

PROOF. Since f is a trilinear form in u, v, w , it is harmonic. We have

$$-L(f) = 2\left(\sum_{i,j=1}^7 f_{u_i} f_{v_j} f_{u_i v_j} + \sum_{i,k=1}^7 f_{u_i} f_{w_k} f_{u_i w_k} + \sum_{j,k=1}^7 f_{v_j} f_{w_k} f_{v_j w_k}\right).$$

By symmetry, it suffices to find the first sum. We have

$$\sum_{i,j=1}^7 f_{u_i} f_{v_j} f_{u_i v_j} = \sum_{i,j=1}^7 \operatorname{Re} o_i(o_v o_w) \operatorname{Re} o_i(o_j o_w) \operatorname{Re} o_u(o_j o_w).$$

Notice that for $\alpha, \beta \in \mathbb{F}_8$

$$\sum_{i,j=1}^7 \operatorname{Re} o_i \alpha \operatorname{Re} o_i \beta = \sum_{i,j=1}^7 \alpha_i \beta_i \equiv \operatorname{Re} \alpha \bar{\beta} - \operatorname{Re} \alpha \operatorname{Re} \beta, \quad (6.31)$$

Hence

$$\sum_{i,j=1}^7 f_{u_i} f_{v_j} f_{u_i v_j} \sum_{j=1}^7 [\operatorname{Re}((o_v o_w) \overline{(o_j o_w)}) - \operatorname{Re} o_v o_w \operatorname{Re} o_j o_w] \operatorname{Re} o_u(o_j o_w) \equiv \Sigma_1 - \Sigma_2$$

We have

$$\Sigma_1 = \sum_{j=1}^7 \operatorname{Re}((o_v o_w)(o_w o_j)) \operatorname{Re} o_u(o_j o_w) \sum_{j=1}^7 \operatorname{Re} o_v(o_w(o_w o_j)) \operatorname{Re} o_u(o_j o_w)$$

Since the octonions are alternative, that is, products involving no more than 2 independent octonions do associate, we find

$$\operatorname{Re} o_v(o_w(o_w o_j)) = \operatorname{Re} o_v((o_w o_w) o_j) = -\operatorname{Re} o_v((o_w \bar{o}_w) o_j) = -w^2 \operatorname{Re} o_v o_j,$$

where $o_w \bar{o}_w = |w|^2 \equiv w^2$ is the norm of the vector w . By (6.27), $\operatorname{Re} o_v o_j = \operatorname{Re} o_v o_j$ and by (6.28), $\operatorname{Re} o_u(o_j o_w) = \operatorname{Re} o_j(o_w o_u)$, so applying (6.31) and (6.30) we obtain

$$\begin{aligned} \Sigma_1 &= -w^2 \sum_{j=1}^7 \operatorname{Re} o_j o_v \operatorname{Re} o_j(o_w o_u) = -w^2 \operatorname{Re} o_v \overline{(o_w o_u)} \\ &= -w^2 \operatorname{Re} o_v(o_u o_w) = w^2 \operatorname{Re} o_v(o_w o_u) = w^2 \operatorname{Re} o_u(o_v o_w) \\ &= w^2 f. \end{aligned}$$

Similarly we obtain

$$\begin{aligned} \Sigma_2 &= \sum_{j=1}^7 \operatorname{Re} o_v o_w \operatorname{Re} o_w o_j \operatorname{Re} o_u(o_j o_w) \sum_{j=1}^7 \operatorname{Re} o_v o_w \operatorname{Re} o_j o_w \operatorname{Re} o_j(o_w o_u) \\ &= \operatorname{Re} o_v o_w \operatorname{Re} o_w \overline{(o_w o_u)} = 0 \end{aligned}$$

because $\operatorname{Re} o_w \overline{(o_w o_u)} = \operatorname{Re} o_w o_u o_w = -w^2 \operatorname{Re} o_u = 0$.

Combining the found identities together, we get

$$L(f) = -2(u^2 + v^2 + w^2)f \equiv -2x^2 f,$$

hence f is a radial eigencubic with $\lambda(f) = -2$. Let us verify that f is indeed an exceptional eigencubic. We have

$$\operatorname{tr} \operatorname{Hess}^2 f = 2 \left(\sum_{i,j=1}^7 f_{u_i v_j}^2 + \sum_{i,k=1}^7 f_{u_i w_k}^2 + \sum_{j,k=1}^7 f_{v_j w_k}^2 \right),$$

where

$$\begin{aligned} \sum_{i,j=1}^7 f_{u_i v_j}^2 &= \sum_{i,j=1}^7 \operatorname{Re} o_i(o_j o_w) \operatorname{Re} o_i(o_j o_w) = \sum_{j=1}^7 (\operatorname{Re}(o_j o_w) \overline{(o_j o_w)} - \operatorname{Re} o_j o_w \operatorname{Re} o_j o_w) \\ &= \sum_{j=1}^7 (\operatorname{Re}(o_j o_w)(o_w o_j) - w_j^2) = 6w^2. \end{aligned}$$

Thus,

$$\operatorname{tr}(\operatorname{Hess} f)^2 = 12x^2. \quad (6.32)$$

By Corollary 5.2, f is an exceptional eigencubic. In view of Table 2 and Proposition 7.4, we have the only possibility: $n_1 = 4$. In order to see this also directly, we observe that by virtue of $\lambda(f) = -2$ and (6.32), $\sigma_2(f) = 6x^2$, so that (5.11) yields $3n_1 + n_2 + 1 = 18$. Since $21 = n = 3n_1 + 2n_2 - 1$, we find $n_1 = 4$ and $n_2 = 5$, as required.

□

6.5. Some remarks. The four Cartan isoparametric eigencubics θ_ℓ are well-known and appear in various mathematical contexts. We mention only a recent interest in θ_ℓ in special Riemannian geometries satisfying the nearly integrability condition [Nu], [GN], explicit solutions to the Ginzburg-Landau system [Fa], [GX], the harmonic analysis of cubic isoparametric minimal hypersurfaces [So1], [So2].

We would like to mention that the octonionic trilinear form discussed in section 6.4 and its quaternionic analogue occur also in calibrated geometries (as an associative calibration on $\text{Im } \mathbb{O}$) [HL] and in constructing of singular solutions of Hessian fully nonlinear elliptic equations [NV1], [NV2].

7. Radial eigencubics and isoparametric quartics

7.1. The degenerate form. By Proposition 3.4, any radial eigencubic having property $n_2 = 0$ is exactly one of the four Cartan's isoparametric eigencubics θ_ℓ . In this section we study non-isoparametric eigencubics, i.e. those with $n_2 \neq 0$. An advantage of working with the degenerate form is that it is more symmetric and can easily be converted into a purely matrix representation. Another important aspect of the degenerate form is that it establishes a correspondence between eigencubics and isoparametric eigencubics which will be studied in the next section.

Proposition 7.1. *Any non-isoparametric radial eigencubic f in \mathbb{R}^n , normalized by $\lambda(f) = -8$, in some orthogonal coordinates has the form*

$$f = (u^2 - v^2)x_n + a(u, w) + b(y, w) + c(u, y, w), \quad (7.1)$$

where $u = (x_1, \dots, x_m)$, $v = (x_{m+1}, \dots, x_{2m})$, $w = (x_{2m+1}, \dots, x_{n-1})$, and the cubic forms $a \in u \otimes w^2$, $b \in v \otimes w^2$, $c \in u \otimes v \otimes w$ satisfy the system

$$\Delta_w a = \Delta_w b = 0, \quad (7.2)$$

$$a_u^2 = b_v^2, \quad (7.3)$$

$$a_w^\top c_w = b_w^\top c_w = 0, \quad (7.4)$$

$$c_v^2 + 2a_w^2 = 4u^2 w^2, \quad (7.5)$$

$$c_u^2 + 2b_w^2 = 4v^2 w^2, \quad (7.6)$$

$$c_u^\top a_{uw} b_w = c_v^\top b_{vw} a_w = 0, \quad (7.7)$$

$$a_u^\top c_{uv} b_v = 0. \quad (7.8)$$

$$2c_w^\top c_{wu} a_u + 2a_w^\top b_{ww} b_w + b_w^\top a_{ww} b_w = 12av^2, \quad (7.9)$$

$$2c_w^\top c_{wv} b_v + 2b_w^\top a_{ww} a_w + a_w^\top b_{ww} a_w = 12bu^2, \quad (7.10)$$

Conversely, any cubic polynomial (7.1) satisfying the above system is a non-isoparametric eigencubic.

PROOF. First notice that in some orthogonal coordinates f is linear with respect to some coordinate function. Indeed, let (3.7) be the normal form of f . Then by our assumption $V_2 \neq \emptyset$ and by (3.12) $(\psi_{030})_\eta^2 = \frac{9}{2}\eta^4$, hence by writing ψ_{030} in the normal form

$$\psi_{030} = \frac{\sqrt{2}}{2}\eta_1^3 + \eta_1\phi(\bar{\eta}) + \psi(\bar{\eta}), \quad \bar{\eta} = (\eta_2, \dots, \eta_{m_2}),$$

we get from (3.7) that

$$f = x_n^3 + \phi x_n + \psi_{030} + \psi_{111} + \psi_{102} + \psi_{012}(x_n + \eta_1\sqrt{2})(x_n - \frac{\eta_1}{\sqrt{2}})^2 + F(x), \quad (7.11)$$

where F is a linear form in the variables η_1 and x_n . Then applying rotation $u = \frac{x_n + \eta_1\sqrt{2}}{\sqrt{3}}$, $v = \frac{x_n\sqrt{2} - \eta_1}{\sqrt{3}}$ in the (η_1, x_n) -plane we conclude that f becomes linear in u in the new coordinates.

Thus, we may assume without loss of generality that f

$$f = x_n \Phi(\bar{x}) + \Psi(\bar{x}), \quad \bar{x} = (x_1, \dots, x_{n-1}). \quad (7.12)$$

Moreover, we shall assume that f is normalized by $\lambda(f) = -8$. Then using (7.12) we obtain from (1.4) by identifying the coefficients of x_n^k

$$\Phi_{\bar{x}}^\top \Phi_{\bar{x}\bar{x}} \Phi_{\bar{x}} = 8\Phi, \quad (7.13)$$

$$2\Phi_{\bar{x}}^\top \Phi_{\bar{x}\bar{x}} \Psi_{\bar{x}} + \Phi_{\bar{x}}^\top \Psi_{\bar{x}\bar{x}} \Phi_{\bar{x}} = 8\Psi, \quad (7.14)$$

$$2\Phi \Phi_{\bar{x}}^2 + 2\Phi_{\bar{x}}^\top \Psi_{\bar{x}\bar{x}} \Psi_{\bar{x}} + \Psi_{\bar{x}}^\top \Phi_{\bar{x}\bar{x}} \Psi_u = 8\bar{x}^2 \Phi, \quad (7.15)$$

$$2\Phi \Phi_{\bar{x}}^\top \Psi_{\bar{x}} + \Psi_{\bar{x}}^\top \Psi_{\bar{x}\bar{x}} \Psi_{\bar{x}} = 8\bar{x}^2 \Psi. \quad (7.16)$$

Since f is harmonic, we also have

$$\Delta \Phi(\bar{x}) = \Delta \Psi(\bar{x}) = 0. \quad (7.17)$$

By virtue of (7.13) we see that the eigenvalues of Φ are ± 1 and 0, and by (7.17) the multiplicities of ± 1 are equal. Denote by $U \oplus V \oplus W$ the associated with Φ eigen decomposition of $\mathbb{R}^n \ominus \text{span}(e_n)$, and let $\bar{x} = u \oplus v \oplus w$ denote the corresponding decomposition of a typical vector $\bar{u} \in V$. Then $\Phi = u^2 - v^2$, and

$$\Psi = \sum_q \Psi_q, \quad \Psi_q \in u^{q_1} \otimes v^{q_2} \otimes w^{q_3},$$

where $q = (q_1, q_2, q_3)$ and $q_i \geq 0$, $q_1 + q_2 + q_3 = 3$. Applying the explicit form of Φ , we obtain from (7.14)

$$\sum_q ((q_1 - q_2)^2 + q_1 + q_2 - 2) \Psi_q = 0$$

and since the non-zero components Ψ_q are linearly independent their coefficients must be zero. This yields

$$\Psi = \Psi_{102} + \Psi_{012} + \Psi_{111} \equiv a + b + c. \quad (7.18)$$

Since $\Delta_{\bar{x}} c = 0$ and a and b are linear in u and v respectively, (7.2) follows from (7.17). Furthermore, (7.15) is equivalent to (7.3)–(7.6). Indeed, we have from (7.18), $\Psi_{\bar{x}}^2 = (a_u + c_u)^2 + (b_v + c_v)^2 + (a_w + b_w + c_w)^2$. By Euler's homogeneity theorem, one readily finds that

$$\Phi_{\bar{x}}^\top \Psi_{\bar{x}\bar{x}} \Psi_{\bar{x}} = 2(b_v^\top c_v - a_u^\top c_u + c_v^2 - c_u^2 + (a_w - b_w)^\top c_w + a_w^2 - b_w^2). \quad (7.19)$$

By using the explicit form of Φ we rewrite (7.15) as

$$\Phi_{\bar{x}}^\top \Psi_{\bar{x}\bar{x}} \Psi_{\bar{x}} + \Psi_u^2 - \Psi_v^2 = 4(u^2 - v^2)w^2,$$

so applying (7.19) we obtain

$$c_v^2 + 2a_w^2 - c_u^2 - 2b_w^2 + 2a_w^\top c_w - 2b_w^\top c_w + a_u^2 - b_v^2 = 4(u^2 - v^2)w^2.$$

Identifying the terms by homogeneity shows that (7.15) is equivalent to (7.3)–(7.6).

We proceed similarly with (7.16). Since the Hessian matrices Ψ_{uu} and Ψ_{vv} are identically zero and $2\Phi \Phi_{\bar{x}}^\top \Psi_{\bar{x}} = 4(u^2 - v^2)(a - b)$, (7.16) becomes

$$\begin{aligned} & 2\Psi_u^\top \Psi_{uv} \Psi_v + 2\Psi_u^\top \Psi_{uw} \Psi_w + 2\Psi_v^\top \Psi_{vw} \Psi_w + \Psi_w^\top \Psi_{ww} \Psi_w \\ & = 4a(u^2 + 3v^2 + 2w^2) + 4b(3u^2 + v^2 + 2w^2) + 8c(u^2 + v^2 + w^2). \end{aligned}$$

By using (7.18) and identifying the terms by homogeneity, we obtain

$$a_u^\top c_{uv} b_v = 0, \quad (7.20)$$

$$2c_u^\top c_{uv} c_v + (a_u^2 + b_v^2)^\top c_w + 2(a_u^\top c_u)^\top a_w + 2(b_v^\top c_v)^\top b_w = 8cw^2, \quad (7.21)$$

$$(a_w^\top b_w)^\top c_w = 0, \quad (7.22)$$

$$c_w a_{ww} c_w = 0, \quad (7.23)$$

$$(a_u^\top c_u)^\top b_w = 0, \quad (7.24)$$

$$(c_v^2 + a_w^2)^\top c_w = 8cu^2, \quad (7.25)$$

$$(2c_v^2 + a_w^2)^\top a_w = 8au^2, \quad (7.26)$$

$$2a_u^\top c_{uv} c_v + (a_u^2 + b_v^2)^\top a_w = 8aw^2, \quad (7.27)$$

$$2(a_w^\top c_w)^\top c_u + 2c_w^\top c_{wu} a_u + 2a_w^\top b_{ww} b_w + b_w^\top a_{ww} b_w = 12av^2, \quad (7.28)$$

and additionally six equations obtained from (7.23)–(7.28) by permutation $(u, a) \leftrightarrow (v, b)$. Then identities (7.23), (7.25), (7.27), (7.21), (7.22) as well as their permutations are corollaries of (7.4)–(7.3). Indeed, (7.22) and (7.23) easily follows from (7.4). Using the first identity in (7.5),

$$(c_v^2 + a_w^2)^\top c_w = (4u^2 w^2 - a_w^2)^\top c_w = 8u^2 c - 2a_w c_{ww} c_w = 8cu^2,$$

so that (7.4) and (7.5) implies (7.25). Similarly,

$$2a_u^\top c_{uv} c_v = a_u^\top (c_v^2)_u = a_u^\top (4u^2 w^2 - 2a_w^2)_u = 8aw^2 - 4a_u a_{uw} a_w,$$

and by virtue of (7.3) we have $(a_u^2 + b_v^2)^\top a_w = 2(a_u^2)^\top a_w = 4a_u^\top a_{uw} a_w$. Thus, (7.27) is a corollary of (7.5) and (7.3). Finally, by virtue of (7.4) and (7.5)

$$c_u^\top c_{uv} c_v + (a_u^2)_w c_w + 2(a_u^\top c_u)^\top a_w \frac{1}{2} c_u^\top (c_v^2 + 2a_w^2)_u + 2a_u^\top (a_w c_w)_u,$$

and after permutation $(u, a) \leftrightarrow (v, b)$

$$c_u^\top c_{uv} c_v + (b_v^2)_w c_w + 2(b_v^\top c_v)^\top b_w = 4cw^2.$$

Summing these identities yields (7.21). Next, (7.20) is equivalent to (7.8) and (7.9) is equivalent to (7.28) modulo (7.4). Finally, (7.7) is equivalent to (7.24) modulo (7.4). For instance,

$$0 = (a_u^\top c_u)^\top b_w = a_u^\top c_{uw} b_w + c_u^\top a_{uw} b_w = a_u^\top (c_u^\top b_w)_u + c_u^\top a_{uw} b_w = c_u^\top a_{uw} b_w.$$

The proposition is proved completely. \square

7.2. A matrix representation of the degenerate form. Now we are going to explain how to extract the type of a non-isoparametric eigencubic from its degenerate representation (7.1). To this end we convert (7.1) into the matrix form

$$f = (u^2 - v^2)x_n + \frac{1}{\sqrt{2}}w^\top A_u w + \frac{1}{\sqrt{2}}w^\top B_v w + 2v^\top M_u w. \quad (7.29)$$

where

$$a = \frac{1}{\sqrt{2}}w^\top A_u w, \quad b = \frac{1}{\sqrt{2}}w^\top B_v w, \quad c = 2v^\top M_u w = 2u^\top N_v w. \quad (7.30)$$

Here $A_u = \sum_{i=1}^m u_i A_i$ etc., and A_i, B_i are symmetric matrices of size $r \times r$, $r = n - 1 - 2m$, and matrices M_i, N_i are of size $m \times r$, $1 \leq i \leq m$. Then (7.2)–(7.6) readily yield

$$\sum_{i=1}^m (w^\top A_i w)^2 = \sum_{i=1}^m (w^\top B_i w)^2, \quad (7.31)$$

$$\text{tr } A_u = \text{tr } B_v = 0, \quad (7.32)$$

$$A_u^3 = u^2 A_u, \quad B_v^3 = v^2 B_v, \quad (7.33)$$

$$\text{tr } A_u^2 B_v = \text{tr } A_u B_v^2 = 0, \quad (7.34)$$

$$A_u^2 + M_u^\top M_u = \mathbf{1}_m, \quad B_v^2 + N_v^\top N_v = \mathbf{1}_m, \quad (7.35)$$

$$A_u M_u^\top = B_v N_v^\top = 0. \quad (7.36)$$

Definition. The representations (7.1) and (7.29) are called the *degenerate forms* of the corresponding radial (non-isoparametric) eigencubic f .

Proposition 7.2. *In the above notation, there exist nonnegative integers p and q such that $2p+q = r$ and for any $u, v \in \mathbb{R}^m$, $u, v \neq 0$, for any i , $1 \leq i \leq m$, the spectrum of any of the matrices $\frac{1}{|u|}A_u$, $\frac{1}{|v|}B_v$, A_i and B_i is $\{\pm 1, 0\}$, where each eigenvalue 1 and -1 has multiplicity p , and the eigenvalue 0 has multiplicity q .*

PROOF. We may assume that $m \geq 1$, otherwise the statement of the proposition is trivial. The assumption $A_u^3 = u^2 A_u$ implies that for any $u \neq 0$, the eigenvalues of A_u are $\pm|u|$ and 0. Denote by $p^\pm(u)$ and $q(u)$ the multiplicities of $\pm|u|$ and 0 respectively. By virtue of (7.32), $\text{tr } A_u = p^+(u) - p^-(u) = 0$. On the other hand, $\text{tr } A_u^2 = p^+(u) + p^-(u)$. The continuous functions $p^+(u)$ and $p^-(u)$ are integer-valued, thus they are identically constant. Denote by p the common value of $p^\pm(u)$. Then $q(u) = r - 2p \equiv q$ also is a constant. Since each matrix A_i is the specialization of A_u when u is the coordinate vector e_i , we conclude that the spectrum of A_i is the same as that of A_u .

Similarly, one defines p' and q' for $\frac{1}{|v|}B_v$, $v \neq 0$. To see that $p = p'$ and $q = q'$, we apply an iterated Laplacian to (7.5). We find by virtue of harmonicity of c and (7.5), $\Delta_v c^2 = 8w^2 u^2 - 4a_w^2$, hence in view of $r = 2p + q$

$$\Delta_w \Delta_v c^2 = 16(2p + q)u^2 - 8 \text{tr } a_{ww}^2 = 16qu^2. \quad (7.37)$$

Applying the u -Laplacian to (7.37) we get

$$\Delta_u \Delta_w \Delta_v c^2 = 32mq.$$

Arguing similarly one finds $\Delta_v \Delta_w \Delta_u c^2 = 32mq'$ which yields $q = q'$. Since $2p + q = 2p' + q'$ we also have $p = p'$. The proposition is proved. \square

Definition. Given a degenerate form of a radial (non-isoparametric) eigencubic, we associate by virtue of Proposition 7.2 a triple of integers (m, p, q) called the *signature* of the degenerate form.

Observe that the type of the degenerate form is determined by virtue of the following formulas which are the corollary of the above definitions:

$$\text{tr } A_u^2 = 2pu^2, \quad \text{tr } B_v^2 = 2pv^2. \quad (7.38)$$

$$\text{tr } M_u^\top M_u \equiv \frac{1}{4} \text{tr } c_{uv} c_{vw} = qu^2, \quad \text{tr } N_v^\top N_v \equiv \frac{1}{4} \text{tr } c_{wu} c_{uw} = qv^2. \quad (7.39)$$

The connection between the signature of the degenerate form and the type of an arbitrary non-isoparametric radial eigencubic is given in the following

Proposition 7.3. *Let f be a non-isoparametric eigencubic of type (n_1, n_2) which has the degenerate form of signature (m, p, q) . Then*

$$n_1 = q, \quad n_2 = m + p + 1 - q, \quad n_3 = m + p - 1 + q. \quad (7.40)$$

PROOF. We only show that $n_1 = q$ because the remaining two identities in (7.40) readily follow from (3.2) and $n = 2m + 2p + q + 1$. To this end, we assume that f is given by (7.1) so that

$$\text{Hess}(f) \begin{pmatrix} f_{x_n x_n} & f_{x_n \bar{x}} \\ f_{\bar{x} x_n} & f_{\bar{x} \bar{x}} \end{pmatrix} \equiv \begin{pmatrix} 0 & (\Phi_{\bar{x}})^\top \\ \Phi_{\bar{x}} & x_n \Phi_{\bar{x} \bar{x}} + \Psi_{\bar{x} \bar{x}} \end{pmatrix},$$

so that

$$\text{tr Hess}^3(f) = x_n^3 \text{tr } \Phi_{\bar{x} \bar{x}}^3 + 3x_n^2 \text{tr } \Phi_{\bar{x} \bar{x}}^2 \Psi_{\bar{x} \bar{x}} + 3x_n (\text{tr } \Phi_{\bar{x} \bar{x}} \Psi_{\bar{x} \bar{x}}^2 + \text{tr } \Phi_{\bar{x}}^\top \Phi_{\bar{x} \bar{x}} \Phi_{\bar{x}}) + \dots$$

where the dots stands for the terms which contain no x_n . Comparing this with (5.12) and (7.1) yields by virtue of $\lambda(f) = -8$ the tautological identities $\text{tr } \Phi_{\bar{x} \bar{x}}^3 = \text{tr } \Phi_{\bar{x} \bar{x}}^2 \Psi_{\bar{x} \bar{x}} = 0$ and additionally

$$\text{tr } \Phi_{\bar{x} \bar{x}} \Psi_{\bar{x} \bar{x}}^2 + \text{tr } \Phi_{\bar{x}}^\top \Phi_{\bar{x} \bar{x}} \Phi_{\bar{x}} = 8(n_1 - 1)(v^2 - u^2). \quad (7.41)$$

In view of $\Phi = u^2 - v^2$, we find $\text{tr } \Phi_{\bar{x}}^\top \Phi_{\bar{x}\bar{x}} \Phi_{\bar{x}} = 8(v^2 - u^2)$. Furthermore,

$$\begin{aligned} \text{tr } a_{wu} a_{uw} &= \frac{1}{2} \Delta_w a_u^2 = \text{by (7.3)} \frac{1}{2} \Delta_w b_v^2 = \text{tr } b_{wv} b_{vw}, \\ \text{tr } a_{wu} c_{uw} &= \text{tr } b_{wv} c_{vw} = \text{by (7.4)} = 0, \\ \text{tr } c_{wu} c_{uw} &= \text{by (7.39)} = 4qv^2. \end{aligned}$$

and taking into account that $\Psi_{uu} = \Psi_{vv} = 0$,

$$\begin{aligned} &\text{tr } \Psi_{\bar{x}\bar{x}} \Phi_{\bar{x}\bar{x}} \Psi_{\bar{x}\bar{x}} 2 \text{tr } (\Psi_{wu} \Psi_{uw} - \Psi_{wv} \Psi_{vw}) \\ &= 2 \text{tr } (a_{wu} + c_{wu})(a_{uw} + c_{uw}) - 2 \text{tr } (b_{wv} + c_{wv})(b_{vw} + c_{vw}) \\ &= 2(\text{tr } c_{wu} c_{uw} - \text{tr } c_{wv} c_{vw}) = -8q(u^2 - v^2). \end{aligned}$$

Substituting the found relations into (7.41) yields $n_1 = q$. \square

In the converse direction, the the signature of the degenerate form of a radial eigencubic is not well determined in general by its type. However, it is well determined, in the most important for us case of exceptional eigencubics.

Corollary 7.1. *Let f be an exceptional eigencubic of type (n_1, n_2) , $n_2 \neq 0$. Then $n_2 = 3\ell + 2$, $\ell \in \{1, 2, 3, 4\}$ and the signature of the degenerate form of f is determined by the formula*

$$(m, p, q) = (\ell + n_1 + 1, 2\ell, n_1). \quad (7.42)$$

PROOF. The first statement follows immediately from Proposition 4.2 and the inequality $n_2 \neq 0$ for non-isoparametric eigencubics. Next, we assume that f is given by (7.29). Then

$$\text{tr Hess}^2(f) = 8mx_n^2 + \dots$$

where the dots stands for the terms with degree of x_n lower than 2. By Proposition 7.1, $\lambda(f) = -8$, hence (5.11) together with $n_2 = 3\ell + 2$ yield $m = n_1 + \ell + 1$. Applying (7.40) we get (7.42). \square

7.3. Proof of Theorem 5. First note a simple corollary of (1.17):

$$h \in \text{Isop}(m_1, m_2) \Leftrightarrow -h \in \text{Isop}(m_2, m_1). \quad (7.43)$$

Let us define two auxiliary polynomials

$$\hat{h}_0(t) = \frac{c_w^2}{4}, \quad \hat{h}_1(t) = \frac{1}{4}(u^2 - v^2)^2 + \frac{c_w^2}{4},$$

which are quartic polynomials in the variable $t = (u, v) \in \mathbb{R}^{2m}$. From (7.6) we obtain

$$\begin{aligned} \frac{1}{4} |\nabla_u |c_w|^2|^2 &= \sum_{i,j,k} c_{w_i} c_{w_i u_k} c_{u_k w_j} c_{w_j} = \frac{1}{2} \sum_{i,j} (c_u^2)_{w_i w_j} c_{w_i} c_{w_j} \\ &= \sum_{i,j} (4u^2 \delta_{ij} c_{w_i} c_{w_j} - 2(b_w^2)_{w_i w_j} c_{w_i} c_{w_j}) \\ &= 4u^2 c_w^2 - 4 \sum_{i,j} b_{w_k w_i} b_{w_k w_j} c_{w_i} c_{w_j} \\ &= 4u^2 c_w^2 - 4 \sum_{i,j} (b_w^\top c_w)_{w_k} (b_w^\top c_w)_{w_k} = 4u^2 c_w^2 \end{aligned}$$

where the last equality is by (7.4). Thus,

$$|\nabla_t \hat{h}_0|^2 = \frac{1}{64} (|\nabla_u |c_w|^2|^2 + |\nabla_v |c_w|^2|^2) = \frac{1}{16} (u^2 + v^2) c_w^2 = 4u^2 \hat{h}_0. \quad (7.44)$$

Since $\Delta_w c^2 = 2c_w^2$, we obtain from (7.37)

$$\Delta_t \hat{h}_0 = \Delta_u \hat{h}_0 + \Delta_v \hat{h}_0 = \frac{1}{8} (\Delta_w \Delta_u c^2 + \Delta_w \Delta_v c^2) = 2qt^2.$$

This implies that h_0 satisfies the Cartan-Münzner equations:

$$|\nabla_u h_0|^2 = 16t^6, \quad \Delta_u h_0 = 8(m+1-2q)t^2,$$

and comparison the latter relation with (1.17) and (1.18) yields the first claimed inclusion.

Now consider h_1 . Define $\widehat{h}_1 = \widehat{h}_0 + \frac{1}{4}(u^2 - v^2)^2$, so that

$$|\nabla \widehat{h}_1|^2 = |\nabla \widehat{h}_0|^2 + 2(u^2 - v^2)(\langle \nabla_u \widehat{h}_0, u \rangle - \langle \nabla_v \widehat{h}_0, v \rangle) + (u^2 - v^2)^2 t^2.$$

Since $c \in u \otimes v \otimes w$ we find

$$\langle \nabla_u \widehat{h}_0, u \rangle = \frac{1}{2} c_w^\top c_{wu} u = \frac{1}{2} c_w^\top c_w = 2h_0,$$

which, in particular, implies $\langle \nabla_u \widehat{h}_0, u \rangle = \langle \nabla_v \widehat{h}_0, v \rangle$. Thus, by (7.44)

$$|\nabla \widehat{h}_1|^2 = |\nabla \widehat{h}_0|^2 + (u^2 - v^2)^2 t^2 = 4t^2 \widehat{h}_0 + (u^2 - v^2)^2 t^2 = 4\widehat{h}_1 t^2.$$

Now it is easy to see that

$$|\nabla_t h_1|^2 = 16t^6, \quad \Delta_t h_1 = 8(m+1-2q)t^2,$$

which yields the second claimed inclusion. Thus the theorem is proved.

7.4. The non-existence results. First combining Corollary 7.1 with Theorem 5 we obtain the following

Corollary 7.2. *To any exceptional eigencubic f of type $(n_1, 3\ell+2)$ one can associate two isoparametric quartic polynomials h_0 and h_1 by virtue of Theorem 5, $h_0 \in \text{Isop}(n_1-1, \ell+1)$ and $h_1 \in \text{Isop}(n_1, \ell)$.*

On the other hand, in [OT1] Ozeki and Takeuchi by using representations of Clifford algebras produced two infinite series of isoparametric families with four principal curvatures. In [FKM], Ferus, Karcher, and Münzner generalized the Ozeiki-Takeuchi approach to produce an infinite family of isoparametric hypersurfaces. More precisely, a quartic polynomial f is said to be of OT-FKM-type [C2, § 4.7] if it is congruent to

$$F_{\mathcal{H}} := t^4 - 2 \sum_{i=0}^s (t^\top H_i t)^2, \quad t \in \mathbb{R}^{2m},$$

where $\mathcal{H} = \{H_0, \dots, H_s\} \in \text{Cliff}(\mathbb{R}^{2m}, s)$. It is easily verified that

$$F_{\mathcal{H}} \in \text{Isop}(s, m-s-1). \quad (7.45)$$

Since the class $\text{Cliff}(\mathbb{R}^{2m}, s)$ is non-empty if and only if

$$s \leq \rho(m), \quad (7.46)$$

the existence of the quartic polynomial (7.45) of OT-FKM-type is equivalent to the inequality

$$\min\{m_1, m_2\} \leq \rho(m_1 + m_2 + 1). \quad (7.47)$$

A culmination of this story is the following deep classification result due to Cecil-Chi-Jensen [CCC] and Immerwoll [Im]: if $h \in \text{Isop}(m_1, m_2)$ and either of the inequalities $m_2 \geq 2m_1 - 1$ or $m_1 \geq 2m_2 - 1$ holds then h is OT-FKM-type.

Proposition 7.4. *There are no exceptional eigencubics of type $(2, 8)$, $(2, 14)$, $(2, 26)$.*

PROOF. We argue by contradiction and suppose that there exist an exceptional eigencubic f of type $(n_1, n_2) \in \{(2, 8), (2, 14), (2, 26)\}$. By Corollary 7.2 we can associate to f an isoparametric quartic polynomial $h_1 \in \text{Isop}(m_1, m_2)$ with $m_1 = n_1$ and $m_2 = \frac{n_2-2}{3}$. Since $n_2 \geq 8$ we have

$$m_2 + 1 - 2m_1 = \frac{n_2 - 2}{3} + 1 - n_1 = \frac{n_2 - 5}{3} \geq 0$$

thus, by Cecil-Chi-Jensen-Immerwoll theorem, h_1 is of OT-FKM-type. This by virtue of (7.47) yields $2 = \min\{m_1, m_2\} \leq \rho(m_1 + m_2 + 1)$. But, for our choice of (n_1, n_2) the number $m_1 + m_2 + 1 = \frac{n_2 + 7}{3}$ is odd, hence $\rho(m_1 + m_2 + 1) = 1$, a contradiction. \square

Proposition 7.5. *There are no exceptional eigencubics of type (3, 8).*

PROOF. Again, we argue by contradiction and suppose that f is an exceptional eigencubic of type (3, 8). By Corollary 7.1, $\ell = 2$, $q = n_1 = 3$ and $m = q + \ell + 1 = 6$. This implies by Corollary 7.2 and (7.43) that the isoparametric quartic polynomials h_i associated to f satisfy

$$h_0 \in \text{Isop}(2, 3), \quad -h_1 \in \text{Isop}(2, 3). \quad (7.48)$$

By Cecil-Chi-Jensen-Immerwoll theorem both h_0 and $-h_1$ are of OT-FKM-type. Hence there exists a Clifford system $\mathcal{H} = \{H_0, \dots, H_s\} \in \text{Cliff}(\mathbb{R}^{2m}, s)$ associated with h_0 and $\mathcal{F} = \{F_0, \dots, F_r\} \in \text{Cliff}(\mathbb{R}^{2m}, r)$ associated with h_1 . In virtue of $m = 6$ the inequality (7.46) yields $s, r \leq 2$. Then we infer from (7.45) and (7.48) that $s = r = 2$. By using (??) we have

$$h_0 \equiv (u^2 + v^2)^2 - 2c_w^2 = t^4 - 2 \sum_{i=0}^2 (t^\top H_i t)^2$$

and

$$-h_1 \equiv u^4 - 6u^2v^2 + v^4 + 2c_w^2 = t^4 - 2 \sum_{j=0}^2 (t^\top F_j t)^2,$$

hence eliminating c_w^2 yields

$$\sum_{i=0}^2 (t^\top H_i t)^2 + \sum_{j=0}^2 (t^\top F_j t)^2 = 4u^2v^2.$$

The latter identity implies the following block forms associated with the decomposition $t = u \oplus v$:

$$H_i = \begin{pmatrix} \mathbf{0} & S_i \\ S_i^\top & \mathbf{0} \end{pmatrix}, \quad F_j = \begin{pmatrix} \mathbf{0} & S_{j+3} \\ S_{j+3}^\top & \mathbf{0} \end{pmatrix}, \quad i, j = 0, 1, 2,$$

thus

$$\sum_{i=0}^5 (u^\top S_i v)^2 = 4u^2v^2, \quad u, v \in \mathbb{R}^6.$$

But the latter identity is a composition formula of size $[6, 6, 6]$ (see, for instance, [Sh]) which contradicts to the Hurwitz theorem stating that a composition formula of size $[k, k, k]$ does exist only if $k = 1, 2, 4, 8$. The contradiction proves the proposition. \square

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